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Some error estimates for solving Volterra integral equations by using the reproducing kernel method

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ABSTRACT

In this paper some error estimates for the reproducing kernel method (RKM) in solving a class of Volterra integral equations are established. Applicability of the estimates are demonstrated by testing some various examples.

1. Introduction

The reproducing kernel method is a promising method which has been applied more and more for solving various problems such as ordinary differential equations, partial differential equations, differential–difference equations, integral equations, etc. in the last decades [1–15]. An approximate solution of the Fredholm integral equation of the first kind in the reproducing kernel space was presented by Du and Cui [6,7], solution of a system of the linear Volterra integral equations was discussed by Yang et al. [8], solvability of a class of Volterra integral equations with weakly singular kernel using RKM was investigated in [9–11], and Geng [12] explained how to solve a Fredholm integral equation of the third kind in the reproducing kernel space. These are a bunch of extensive works related to RKM for solving integral equations.

Recently, some authors have tried to find error estimates of the reproducing kernel method. For example, Geng [12] prepared an error estimate for the method in solving integral equations of the third kind, Lin and Wu [13] found some error estimates of the method in solving a linear second-order two-point boundary value problem and Geng et al. [14] reported some error estimates of the method in solving a singularly perturbed turning point problem with an interior layer.

In this paper we aim to establish some error estimates of the reproducing kernel method in solving a class of Volterra integral equations which has not yet been discussed according to the knowledge of the authors. For this purpose, we consider the following nonlinear Volterra integral equation

$$u(x) = F(x, u(x)), \tag{1}$$

where

$$F(x, u(x)) = f(x) + \int_0^x k(x, \xi)\mathcal{N}(u(\xi))d\xi, \quad x \in [0, 1],$$

in which functions f and k and the nonlinear operator $\mathcal{N}$ are considered such that Eq. (1) has a unique solution.

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The rest of the paper is organized as follows. In the next section, the reproducing kernel method for solving Eq. (1) is presented. Error estimates are established in Section 3. Some numerical examples are provided in Section 4 for confirming the theoretical results. Section 5 is a brief conclusion.

2. Reproducing kernel method

In this section, we follow the recent work of Cui et al. [15] and represent some useful materials.

Definition 1. For a natural number m the function space $W_m[0, 1]$ is defined as follows

$$W_m[0, 1] = \{u(x) | u^{(m-1)}(x) \text{ is an absolutely continuous function and } u^{(m)}(x) \in L^2[0, 1]\}.$$

The inner product and norm in $W_m[0, 1]$ are defined respectively by

$$\langle u, v\rangle_{W_m} = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x)dx, \quad \forall u, v \in W_m[0, 1],$$

and

$$\|u\|_{W_m} = \sqrt{\langle u, u\rangle_{W_m}}, \quad \forall u \in W_m[0, 1].$$

In general, the function space $W_m[0, 1]$ is a reproducing kernel space and its reproducing kernel $R_m(x, y)$ has the following reproducing property

$$u(x) = \langle u(.), R_m(., x)\rangle_{W_m}, \quad \forall u \in W_m[0, 1]. \tag{2}$$

The function spaces $W_1[0, 1]$ and $W_2[0, 1]$ are reproducing kernel spaces and their reproducing kernels are respectively

$$R_1(x, y) = \begin{cases} 1 + x, & x \le y, \\ 1 + y, & x > y, \end{cases}$$

and

$$R_2(x, y) = \begin{cases} 1 + yx + \frac{1}{2}yx^2 - \frac{1}{6}x^3, & x \le y, \\ 1 + xy + \frac{1}{2}xy^2 - \frac{1}{6}y^3, & x > y, \end{cases}$$

see e.g. [15] and Appendix for details. For $m = 1, 2$, putting $\psi_i(x) = R_m(x, x_i)$, where $\{x_i\}_{i=1}^{\infty}$ is a set of points in the interval $[0, 1]$, the orthonormal system $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ can be derived from $\{\psi_i(x)\}_{i=1}^{\infty}$ by the Gram–Schmidt orthogonalization process, i.e.,

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0,\ i = 1, 2, \ldots).$$

Theorem 1. *Let $\{x_i\}_{i=1}^{\infty}$ be dense in the interval $[0, 1]$. If Eq. (1) has a unique solution, then it can be represented in $W_m[0, 1]$ as follows:*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik}F(x_k, u(x_k))\bar{\psi}_i(x). \tag{3}$$

We rewrite (3) as

$$u(x) = \sum_{j=1}^{\infty} A_j\bar{\psi}_j(x),$$

where

$$A_j = \sum_{l=1}^{j} \beta_{jl}F(x_l, u(x_l)).$$

Now, an approximate solution can be obtained by taking finitely many terms in the series representation of u as

$$u_{mN}(x) = \sum_{j=1}^{N} B_j\bar{\psi}_j(x),$$

where $B_j = \sum_{l=1}^{j} \beta_{jl}F(x_l, u_{m,j-1}(x_l))$. Obviously, $u_{mN} \in W_m[0, 1]$. In the following algorithm, we summarize how the method works.