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A comparative study on the weak Galerkin, discontinuous Galerkin, and mixed finite element methods

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ABSTRACT

This paper presents a comparative study on the newly introduced weak Galerkin finite element methods (WGFEMs) with the widely accepted discontinuous Galerkin finite element methods (DGFEMs) and the classical mixed finite element methods (MFEMs) for solving second-order elliptic boundary value problems. We examine the differences, similarities, and connection among these methods in scheme formulations, implementation strategies, accuracy, and computational cost. The comparison and numerical experiments demonstrate that WGFEMs are viable alternatives to MFEMs and hold some advantages over DGFEMs, due to their properties of local conservation, normal flux continuity, no need for penalty factor, and definiteness of discrete linear systems.

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1. Introduction

For convenience of presentation, we concentrate on two-dimensional elliptic boundary value problems formulated as

$$\begin{cases} \nabla\cdot(-\mathbf{K}\nabla p) \equiv \nabla\cdot\mathbf{u} = f, & \mathbf{x}\in\Omega, \\ p = p_D, & \mathbf{x}\in\Gamma^D, \\ \mathbf{u}\cdot\mathbf{n} = u_N, & \mathbf{x}\in\Gamma^N, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded polygonal domain, p the primal unknown, $\mathbf{K}$ a conductivity or permeability tensor that is uniformly symmetric positive-definite, f a source-term, p_D, u_N respectively Dirichlet and Neumann boundary data, $\mathbf{n}$ the unit outward normal vector on $\partial\Omega$, which has a non-overlapping decomposition $\Gamma^D\cup\Gamma^N$.

When $\Gamma^D\neq\emptyset$, the problem has a unique solution. When $\Gamma^D=\emptyset$, a consistency condition

$$\int_\Omega f d\mathbf{x} = \int_{\Gamma^N} u_N ds$$

is specified to ensure uniqueness of the solution.

The model problem (1) arises from many practical problems, for example, flow in porous media, heat or electrical conduction in composite materials. In the context of porous medium flow, p is the pressure for a single-phase steady flow, $\mathbf{K}$ is the ratio of permeability and fluid viscosity, and $\mathbf{u} = -\mathbf{K}\nabla p$ is the Darcy velocity. In the context of heat (electric)

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conduction, p is the temperature (electric potential), $\mathbf{K}$ is the thermal (electric) conductivity, and $\mathbf{u} = -\mathbf{K}\nabla p$ is heat (electric) flux. All these applications call for accurate, efficient, and robust numerical approximations of not only the primal variable (pressure, temperature, or electrical potential) but also the flux (Darcy velocity, heat or electrical flux).

There have been a variety of numerical methods for the model problem (1): the continuous Galerkin finite element methods (CGFEMs), the DGFEMs, and the MFEMs, in addition to the finite difference methods and finite volume methods. All these numerical methods result in large-scale discrete linear systems, which are solved directly or iteratively. Besides accuracy, efficiency, and robustness of numerical methods, physical properties such as local conservation and flux continuity are also major concerns in practical applications mentioned above.

One could say there are already plenty of numerical methods for even just a simple elliptic boundary value problem like (1). *Is there any need for developing new numerical methods for an already well studied model problem? What is it good for with the new WGFEMs?*

To answer the above questions, let us briefly examine the main features of the existing numerical methods.

(i) The CGFEMs are known as lacking of "local conservation", even though they are conceptually simple and have relatively less unknowns [1,2].
(ii) The DGFEMs are locally conservative but *there is no continuity in the DG flux* [3]. DGFEMs have flexibility in handling complicated geometry but proliferate in numbers of unknowns. Choosing problem-dependent penalty factors is a drawback for practical use of DGFEMs.
(iii) The MFEMs approximate the primal variable and flux simultaneously using finite element pairs that satisfy the inf–sup condition. But the resulting indefinite linear systems (saddle-point problems) require special solvers [4,5].

The weak Galerkin finite element methods introduced in [6] adopt a completely different approach. They rely on novel concepts such as the weak gradient and discrete weak gradients. As is well known, the variational formulation of a second-order elliptic problem relies on the duality of the classical gradient operator. Locality of a finite element space implies local conservation and relative independence of elementwise shape functions. However, shape functions in element interiors are related to their values on the element interfaces (through integration by parts). The weak gradient operator characterizes exactly this connection, see Eq. (12) in Section 3. Discrete weak gradients inherit the connection for finite dimensional (Galerkin type) polynomial approximation subspaces, see Eq. (14) in Section 3.

As novel WGFEMs are being developed for different types of problems, e.g., second order elliptic problems, bi-harmonic problems, and Stokes flow, there arises a need for comparing WGFEMs with the existing numerical methods, especially the classical mixed finite element methods and the intensively investigated DGFEMs. This paper addresses such a need by providing fair comparison of these three types of methods. We compare these methods on scheme formulation, accuracy and error estimates, numbers of unknowns and condition numbers, implementation issues, and desired physical properties.

The rest of this paper is organized as follows. Section 2 presents preliminaries about RT_0, BDM_1 finite elements that are common to MFEMs, WGFEMs, and DGFEMs postprocessing, especially for flux calculations. Section 3 presents WGFEMs, DGFEMs, MFEMs and their implementations. Section 4 presents detailed comparison of WGFEMs with DGFEMs and MFEMs. Section 5 presents numerical results to further examine the differences among these methods. Section 6 concludes the paper with some remarks.

2. Preliminaries: bases for *RT*, *BDM* finite element spaces

The divergence form of the second order elliptic problem in (1) indicates the importance of the space $H(\text{div}, \Omega)$ and $H(\text{div})$-conforming finite element spaces. We define

$$H(\text{div}; \Omega) = \left\{\mathbf{v} \in L^2(\Omega)^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega)\right\},$$
$$H_{0,N}(\text{div}; \Omega) = \left\{\mathbf{v} \in H(\text{div}; \Omega) : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma^N\right\},$$
$$H_{u_N,N}(\text{div}; \Omega) = \left\{\mathbf{v} \in H(\text{div}; \Omega) : \mathbf{v} \cdot \mathbf{n} = u_N \text{ on } \Gamma^N\right\}.$$

Raviart–Thomas (RT) elements and Brezzi–Douglas–Marini (BDM) elements are among the most frequently used [7,3,5,8,9,6]. In this section, we briefly discuss some interesting properties of the basis functions for these elements.

2.1. Edge-based bases for RT_0 and BDM_1

Barycentric coordinates are enjoyed by practitioners of FEMs. Let $T = \Delta P_1P_2P_3$ be a triangle oriented counterclockwise and $|T|$ be its area. For any point $P(x, y)$ on the triangle, let $|T_i|(i = 1, 2, 3)$ be the areas of the small triangles when $P_i(x_i, y_i)$ are respectively replaced by P (see Fig. 1). Then $\lambda_i = |T_i|/|T|(i = 1, 2, 3)$ are the barycentric coordinates. Clearly $0 \le \lambda_i \le 1$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

It is clear that $\lambda_i(x, y)(i = 1, 2, 3)$ are also the Lagrangian P1 basis functions, whose gradients are

$$\begin{aligned}\nabla\lambda_1 &= \langle y_2 - y_3, x_3 - x_2\rangle/(2|T|),\\ \nabla\lambda_2 &= \langle y_3 - y_1, x_1 - x_3\rangle/(2|T|),\\ \nabla\lambda_3 &= \langle y_1 - y_2, x_2 - x_1\rangle/(2|T|).\end{aligned} \tag{2}$$

Their integrals are nicely expressed in the following lemma.