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Multiple coarse grid acceleration for multiscale multigrid computation

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ABSTRACT

The multiscale multigrid method uses an iterative refinement procedure with the Richardson extrapolation technique to obtain a higher order solution. The computational cost for the iterative refinement procedure may be significant for some ill conditioned coefficient matrices. In this paper, we proposed an alternative strategy using multiple coarse grids to eliminate the iterative refinement procedure, and thus accelerate the multiscale multigrid computation. Numerical investigations show that our multiple coarse grid computing strategy is more efficient and scalable than the iterative refinement procedure. The multiscale multigrid method with multiple coarse grid strategy is used to solve two dimensional (2D) Poisson equation and convection diffusion equation, but the idea can be used to solve other partial differential equations.

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1. Introduction

In the field of finite difference method for solving partial differential equations, there are growing interests in developing explicit high order compact schemes [1–6]. Compared with lower order schemes [7,8], higher order difference schemes are preferred in large scale simulations and modeling applications because they usually yield better computed solutions with less computational cost. Here "compact" means that these schemes only use the center node and the adjacent nodes in each dimension so that they can process boundary points effectively. The word "explicit" means that these methods compute the solution of variables directly and do not need to compute the derivatives of variables.

Recently, Zhang et al. proposed a series of explicit methods for sixth order compact computations by using Richardson extrapolation [9–12]. Among these methods, the most efficient one is called the multiscale multigrid (MSMG) method, which incorporates the sixth order explicit compact computing strategy and multigrid solution idea [10]. Compared with traditional multigrid methods, the MSMG has more advantages in computing higher order solution. On one hand, the MSMG method inherits the merit from the standard multigrid method offering the convergence rate that is independent of the grid size [13,14]. On the other hand, the existing multilevel structure of the standard multigrid method can be used in the multiscale computing technique in a seamless way to improve the solution accuracy.

In the existing MSMG computational framework, the Richardson extrapolation technique is used to compute a sixth order solution on the standard coarse grid after obtaining fourth order solutions on the fine and coarse grids. Then, an

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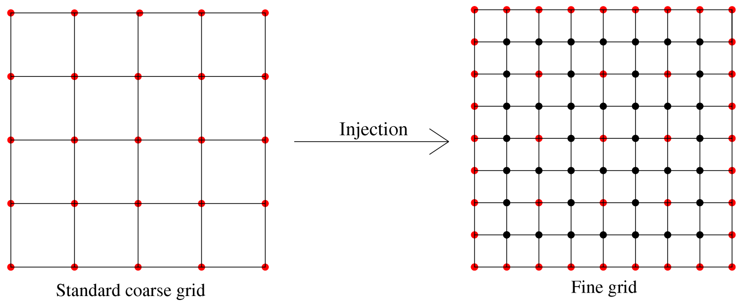

Fig. 1. Injection from the standard coarse grid to the fine grid. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

interpolation technique is needed to interpolate the sixth order solution from the standard coarse grid to the fine grid. In 2D, for the $(even, even)$ indexed fine grid points, the solution can be directly interpolated (injected) from the standard coarse grid and it can keep the sixth order accuracy, as in Fig. 1. In Fig. 1, the grid points marked in red have sixth order solutions, while the grid points marked in black have fourth order solutions. In order to improve the accuracy of the solution of black fine grid points, Wang and Zhang [10] used an operator based interpolation scheme to update them in a specific sequence iteratively. However, their interpolation scheme may converge slowly for some convection diffusion equations with high Reynolds number. In this paper, our goal is to reduce the computational cost by using an alternative method to update the black fine grid points in Fig. 1. Therefore, we develop a computing strategy using multiple coarse grids to directly calculate higher order solutions and thus accelerate the MSMG computation by eliminating the iterative refinement procedure.

2. Multiple coarse grid computing strategy

The idea of multiple coarse grid (MCG) computation can be traced back to the parallel superconvergent multigrid method [15]. In general, for a d dimensional problem, the fine grid can easily be coarsened into 2^d coarse grids. Therefore, four coarse grids are generated from the fine grid in 2D, as in Fig. 2.

In Fig. 2, the (odd, odd) coarse grid consists of (odd, odd) indexed grid points (green-colored) from the fine grid; the $(odd, even)$ coarse grid consists of $(odd, even)$ indexed grid points (black-colored) from the fine grid; the $(even, odd)$ coarse grid consists of $(even, odd)$ indexed grid points (blue-colored) from the fine grid; the $(even, even)$ coarse grid is the standard coarse grid, which consists of $(even, even)$ indexed grid points (red-colored) from the fine grid. In addition, all boundary points are marked in red.

In order to illustrate the MCG computing strategy, we consider the 2D Poisson equation of the form

$$u_{xx}(x, y) + u_{yy}(x, y) = f(x, y), \quad (x, y) \in \Omega, \tag{1}$$

where Ω is a rectangular domain $[0, L_x] \times [0, L_y]$ with appropriate boundary conditions defined on $\partial\Omega$. Ω is discretized with uniform mesh sizes $\Delta x = L_x/N_x$ and $\Delta y = L_y/N_y$ in the x and y coordinate directions, respectively. Here N_x and N_y are the number of uniform intervals in the x and y coordinate directions, respectively. The mesh points are (x_i, y_j) with $x_i = i\Delta x$ and $y_j = j\Delta y$, $0 \le i \le N_x$, $0 \le j \le N_y$. The solution $u(x, y)$ and the forcing function $f(x, y)$ are assumed to be sufficiently smooth and have required continuous partial derivatives.

A general fourth order compact (FOC) scheme for Eq. (1) with the mesh aspect ratio $\lambda = \Delta x/\Delta y$ has the following form [6]

$$\begin{aligned} &d(u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1}) + c(u_{i,j+1} + u_{i,j-1}) + b(u_{i+1,j} + u_{i-1,j}) - au_{i,j} \\ &\quad = \frac{\Delta x^2}{2}(8f_{i,j} + f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}), \end{aligned} \tag{2}$$

where the coefficients are

$$a = 10(1 + \lambda^2), \qquad b = 5 - \lambda^2, \qquad c = 5\lambda^2 - 1, \qquad d = (1 + \lambda^2)/2.$$

After using Eq. (2) on Ω_Δ and $\Omega_{2\Delta}$, both fine and standard coarse grids have computed fourth order solutions. Then, the Richardson extrapolation technique can be applied to compute a higher order solution. The general Richardson extrapolation can be written as [16]

$$\tilde{u}^{2h}_{i,j} = \frac{(2^p u^h_{2i,2j} - u^{2h}_{i,j})}{2^p - 1}, \tag{3}$$

where p is the order of accuracy before the extrapolation, and the order of accuracy will be increased to $p + 2$ after the extrapolation. If the order p is equal to four, the sixth order accurate solution on the standard coarse grid can be computed by the formula

$$\tilde{u}^{2h}_{i,j} = \frac{(16u^h_{2i,2j} - u^{2h}_{i,j})}{15}. \tag{4}$$