Journal of Computational and Applied Mathematics 269 (2014) 132–149

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A posteriori error estimates of stabilized low-order mixed finite elements for the Stokes eigenvalue problem

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ARTICLE INFO

Article history:
Received 2 August 2012
Received in revised form 28 November 2013

MSC:
65N12
65N25
65N30

Keywords:
Stokes eigenvalue problem
Stabilized mixed methods
A posteriori error estimates

ABSTRACT

In this paper we obtain a priori and a posteriori error estimates for stabilized low-order mixed finite element methods for the Stokes eigenvalue problem. We prove the convergence of the method and a priori error estimates for the eigenfunctions and the eigenvalues. We define an error estimator of the residual type which can be computed locally from the approximate eigenpair and we prove that, up to higher order terms, the estimator is equivalent to the energy norm of the error. We also present some numerical tests which show the performance of the adaptive scheme.

1. Introduction

Adaptive procedures based on a posteriori error estimators have gained an enormous importance in the numerical approximation of partial differential equations. Several approaches, most of them focused on source problems, have been considered to construct estimators based on the residual equations (see [1,2] and their references). Moreover, for the standard Laplace eigenvalue problem a simple and clear analysis has been obtained in [3,4], and there are some similar results for other eigenvalue problems (see, for example, [5–8] and the references therein). However, there are few results concerning a posteriori error estimates for the Stokes eigenvalue problem. In [9] the authors present an a posteriori error analysis for the Stokes eigenvalue problem assuming that the schemes used in its finite element discretization are stable (as, for example, the mini elements).

Despite the fact that the lower-order mixed finite elements for the Stokes equations violate the inf–sup condition, it is well known that low-order velocity–pressure pairs have a relevant interest due to its simple and attractive computational aspects (see [10] and the references therein). There are many stabilized finite element methods to counteract the lack of stability (see, for example, [11–17]). In particular, Bochev, Dohrmann and Gunzburger proposed in [13] a new family of stabilized methods, for the source Stokes problem, and proved that this simple and useful approach is unconditionally stable. Based on this work, in [18] the authors introduce an a posteriori error indicator, for the source Stokes problem, and it yields global upper and lower bounds on the error of stabilized finite element methods.

In this work we prove the convergence of stabilized low-order mixed finite elements for the Stokes eigenvalue problem and we obtain optimal a priori error estimates for the eigenfunctions and the eigenvalues by using the spectral theory given

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http://dx.doi.org/10.1016/j.cam.2014.03.027
0377-0427/

in [19]. We define an a posteriori error estimator of the residual type which can be computed locally from the approximate eigenpair. We show its global reliability and local efficiency by proving that the estimator is equivalent to the energy norm of the error up to higher order terms. We also present some numerical tests which allow us to show the good performance of the error indicator and the adaptive algorithm.

The rest of the paper is organized as follows. In Section 2 we introduce the Stokes eigenvalue problem. In Section 3 we present the stabilized low-order mixed finite element method and obtain L^2 a priori error estimates. In Section 4 we prove the convergence for the eigenfunctions and the eigenvalues. In Section 5 we introduce the a posteriori error estimator and prove its equivalence with the energy norm of the error. In Section 6 we report some numerical examples which allow us to assess the performance of the adaptive scheme.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and polygonal domain with boundary $\Gamma := \partial\Omega$. For $\mu \geq 0$ we consider the Stokes eigenvalue problem: Find $(\mathbf{u}, p, \lambda)$, with $\mathbf{u} = (u_1, u_2) \neq 0$ and $\lambda \in \mathbb{R}$, such that

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \lambda\mathbf{u} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

which models the slow motion of an incompressible viscous fluid occupying Ω, where $\mathbf{u}$ is the fluid velocity and p is the pressure.

We will denote by boldface the spaces consisting of vector value functions. Let $\mathbf{V} := \mathbf{H}_0^1(\Omega)$ and $S := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$. The norms and seminorms in $\mathbf{H}^m(D)$, with m an integer, are denoted by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ respectively and $(\cdot,\cdot)_D$ denotes the inner product in $L^2(D)$ or $\mathbf{L}^2(D)$ for any subdomain $D \subset \Omega$. The domain subscript is dropped for the case $D = \Omega$.

Problem (1) can be written, after normalization for $\mathbf{u}$, in a variational form as follows:

Find $(\mathbf{u}, p, \lambda) \in (\mathbf{V}, S, \mathbb{R})$, with $\|\mathbf{u}\|_0 = 1$, such that

$$Q(\mathbf{u}, p, \mathbf{v}, q) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall(\mathbf{v}, q) \in (\mathbf{V}, S), \tag{2}$$

where

$$Q(\mathbf{u}, p, \mathbf{v}, q) = \mu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} - \int_\Omega p\nabla\cdot\mathbf{v} - \int_\Omega q\nabla\cdot\mathbf{u},$$

with $(\cdot,\cdot)$ the inner product in $\mathbf{L}^2(\Omega)$.

Now, it is clear that the symmetric bilinear form Q is continuous, i.e., for every $(\mathbf{u}, q), (\mathbf{v}, s) \in (\mathbf{V}, S)$

$$Q(\mathbf{u}, q, \mathbf{v}, s) \leq C(\|\mathbf{u}\|_1 + \|q\|_0)(\|\mathbf{v}\|_1 + \|s\|_0),$$

moreover it is known [20,21] that $Q(\mathbf{u}, q, \mathbf{v}, s)$ satisfies the following inf–sup condition with a positive constant β:

$$\sup_{(\mathbf{v},s)\in(\mathbf{V},S)} \frac{Q(\mathbf{u}, q, \mathbf{v}, s)}{\|\mathbf{v}\|_1 + \|s\|_0} \geq \beta(\|\mathbf{u}\|_1 + \|q\|_0) \quad \forall(\mathbf{u}, q) \in (\mathbf{V}, S), \tag{3}$$

and so the bilinear form Q is stable.

Now, from the spectral theory (see [19]) we know that the eigenvalue problem (2) has a positive eigenvalue sequence λ_j which we assume to be increasingly ordered:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \lim_{j\to+\infty} \lambda_j = +\infty,$$

and the associated eigenfunctions

$$(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2), \ldots, (\mathbf{u}_k, p_k), \ldots$$

with $(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}$. For simplicity, we only consider simple eigenvalues in this paper.

3. Stabilized mixed finite element approximations

Let $\mathcal{T}_h$ be a family of triangulations of Ω such that any two triangles in $\mathcal{T}_h$ share at most a vertex or an edge. Let h stand for the mesh-size; namely $h = \max_{T\in\mathcal{T}_h} h_T$, with h_T being the diameter of the triangle T. We assume that the family of triangulations $\{\mathcal{T}_h\}$ satisfies a minimum angle condition, i.e., there exists a constant $\tau > 0$ such that $h_T/r_T \leq \tau$, where r_T is the diameter of the largest circle contained in T.

Let

$$P_1(\Omega) = \{u \in C(\Omega) |\ u|_T \in \mathcal{P}_1(T)\ \forall\, T \in \mathcal{T}_h\}.$$

We consider the pair

$$\mathbf{V}^h = \mathbf{P}_1 \cap \mathbf{H}_0^1(\Omega) \quad \text{and} \quad S^h = P_1 \cap L_0^2(\Omega). \tag{4}$$