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# Dynamic programming for a Markov-switching jump–diffusion

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ABSTRACT

We consider an optimal control problem with a deterministic finite horizon and state variable dynamics given by a Markov-switching jump–diffusion stochastic differential equation. Our main results extend the dynamic programming technique to this larger family of stochastic optimal control problems. More specifically, we provide a detailed proof of Bellman's optimality principle (or dynamic programming principle) and obtain the corresponding Hamilton–Jacobi–Belman equation, which turns out to be a partial integro-differential equation due to the extra terms arising from the Lévy process and the Markov process. As an application of our results, we study a finite horizon consumption–investment problem for a jump–diffusion financial market consisting of one risk-free asset and one risky asset whose coefficients are assumed to depend on the state of a continuous time finite state Markov process. We provide a detailed study of the optimal strategies for this problem, for the economically relevant families of power utilities and logarithmic utilities.



## 1. Introduction

When speaking about the properties of real life as it expresses itself in nature, we learn that it is nonlinear rather than linear. One way to move between both is piecewise linearity, which we can easily generalize to piecewiseness or "hybridicity" in general, e.g., in Engineering, specifically in Electrical Engineering and Electronics, and in Economics. But we also know that real life is, to some extent, discontinuous rather than continuous, e.g., in processes of Biology, Medicine, Engineering and Finance, so that phenomena of impulsiveness or "jumps" need to be taken into consideration, too. In a further step, we may allow both generalizations together in the sense that "regime" switches and jumps can occur randomly and, additionally, that the dynamics is stochastic in both the state space, that now permits jumps, and in some discrete space of states, which tell us in which discrete "mode" we are. An appropriate way to express this dynamics is stochastic hybrid systems with jumps, as represented by a stochastic differential equation (SDE) possibly equipped with conditional transition probabilities, or by a system of SDEs.

In this paper, we consider a decision making, or optimal control problem, subject to an underlying stochastic hybrid system with jumps. These problems are both relevant from the practical point of view and challenging mathematically (see,

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e.g., [1] and references therein for further details). To have entered the areas of finance and insurance in the presence of stochastic hybrid systems with jumps herewith is a core achievement of those works and of ours. The time-continuous model in financial mathematics and actuarial sciences, expressed as portfolio optimization or, dually to that maximization, the minimization of expected costs, under finite maturity time (''finite horizon''), usually follows one of the following approaches: martingale duality methods, consisting of a static optimization problem and a representation problem, or stochastic control, consisting of a parametric optimization problem in the (deterministic) control space followed by a partial differential equation. In the latter approach, the necessary optimality conditions provided by the dynamic programming principle or the Hamilton–Jacobi–Bellman (HJB) equation need to be addressed. Herein, we are very close to Bellman's dynamic programming technique, which we translate into our hybrid setting with jumps.

At this point we should mention that the dynamic programming technique was firstly introduced by Richard Bellman in the 1950s to deal with calculus of variations and optimal control problems [2–5]. Further developments have been obtained since then by a number of scholars including Florentin [6,7] and Kushner [8], among others. The approach introduced by Bellman relies on the description of the value function associated with a given optimal control problem through a backwards recursive relation, known currently as Bellman's optimality principle. Under additional regularity conditions, it can be proved that such a value function is also the solution of a partial differential equation, known as Hamilton–Jacobi–Bellman equation. A very complete treatment of the modern theory of optimal control problems can be found in the excellent monographs by Fleming and Soner [9], Yong and Zhou [10] and Oksendal and Sulem [11].

In the present paper, we demonstrate a dynamic programming principle for an optimal control problem with finite deterministic horizon and state variable dynamics given by a Markov-switching jump–diffusion stochastic differential equation. Moreover, we find the associated Hamilton–Jacobi–Bellman (HJB) equation, which in our case is a partial integro-differential equation due to the extra terms arising from the Lévy process terms and the Markov process driving the switching. The approach just described is distinct from the one of followed in [1]. The later paper introduces a numerical approach which is a multiple and prosperous extension of the one introduced by Koutsoukos in [12], concerning the aforementioned field, together with an application. Our contribution comes less from the numerical point of view, and is more centred on the theoretical framework, being of a widely analytical nature.

As an application of the abstract results presented here, we investigate a consumption–investment problem in a jump–diffusion financial market consisting of one risk-free and one risky asset whose coefficients are supposed to depend on the state of a continuous time finite state Markov process. Here, we present a detailed investigation of the optimal strategies, for power utilities and logarithmic utilities as well. The consumption–investment problem was firstly studied by Merton in his seminal papers [13,14]. This problem has been thoroughly studied ever since, including extensions to jump–diffusion financial markets (see, e.g., the series of papers by Framstad et al. [15–17] and references therein). In what concerns Markov-switching behaviour in economics and finance, this has been considered by Hamilton in [18] to explain shifts in growth rates of Gross National Product (GNP), Elliott et al. in [19,20] to address problems related with option pricing and risk minimization, Zhang in [21] to determine optimal selling rules and by Zhang and Yin to deal with optimal asset allocation rules [22]. To the best of our knowledge, dynamic programming techniques have not yet been applied to the consumption–investment problem with an underlying Markov-switching jump–diffusion financial market.

We strongly believe that the methods and techniques developed here may be of interest to a wide range of topics in Applied Science, Computing and Engineering, eventually leading to future integration and comparison with other heuristic and model-free approaches and methods (see, e.g., [23–28] and references therein).

This paper is organized as follows. In Section 2, we describe the setting we work with and formulate the problem we propose to address. Section 3 contains the dynamic programming principle and the HJB partial integro-differential equations, as well as the corresponding verification theorem and its proof. In Section 4, we address a consumption–investment problem and study the particular case of power utility functions and logarithmic utility functions. We conclude in Section 5.

## 2. Setup and problem formulation

Let $T > 0$ be a deterministic finite horizon and let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ satisfying the usual conditions, i.e., $\mathbb{F}$ is an increasing, right-continuous filtration and $\mathcal{F}_0$ contains all $P$-null sets. For each $d \in \mathbb{N}$, let $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$ and let $\mathcal{B}_0^d$ be the Borel $\sigma$-field generated by the open subsets $O$ of $\mathbb{R}_0^d$ whose closure does not contain 0.

We will consider the following stochastic processes throughout this paper:

(i) a standard $M$-dimensional Brownian motion $W(t) = \{W(t) : t \in [0, T]\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

(ii) a continuous time Markov process $\{\alpha(t) : t \in [0, T]\}$ with a finite space state $S = \{a_1, \dots, a_n\}$ and generator $Q = (q_{ij})_{i,j \in S}$. Let $N_{ij}(t)$ denote the counting process given by

$$N_{ij}(t) = \sum_{0<s\leq t} I_{\{\alpha(s_-)=i\}} I_{\{\alpha(s)=j\}},$$

where $I_A$ denotes the indicator function of a set $A$. Note that $N_{ij}(t)$ gives the number of jumps of the Markov process $\alpha$ from state $i$ to state $j$ up to time $t$. Define the intensity process by

$$\lambda_{ij}(t) = q_{ij} I_{\{\alpha(t_-)=i\}},$$