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# Properties of generators of quasi-interpolation operators of high approximation orders in spaces of polyharmonic splines

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ABSTRACT

We have presented in Bozzini et al. (2011) a procedure in spaces of $m$-harmonic splines in $\mathbb{R}^d$ that starts from a simple generator $\phi_0$ and recursively defines generators $\phi_1, \phi_2, \ldots, \phi_{m-1}$ with corresponding quasi-interpolation operators reproducing polynomials of degrees $3, 5, \ldots, 2m-1$ respectively. In this paper we study the properties of generators $\phi_j$, and we prove that these new generators are positive definite functions, and are scaling functions whenever $\phi_0$ has those properties. Moreover $\phi_0$ and $\phi_j$ generate the same multiresolution analysis. We show that it is possible to define a convergent subdivision scheme, and to provide in this way a fast computation of the quasi-interpolant.



## 1. Introduction

Polyharmonic splines have been introduced in [1] as a variational extension to many variables of polynomial splines in one variable. Because of their good properties (see e.g. [2–4]) they are studied by several authors, and widely used in scattered data approximation and interpolation. An important amount of research work has been done also on regular grids which occur in many applications such as image processing and multiresolution analysis. In general it is important to construct quasi-interpolation operators capable of reproducing some important functions of the space, and to provide a fast evaluation of the quasi-interpolant function.

The space of $m$-harmonic splines $V_{2m}$, $2m > d$, is defined as the subspace of $S'(\mathbb{R}^d)$ (the space of $d$-dimensional tempered distributions)

$$V_{2m} = \left\{ g \in S'(\mathbb{R}^d) \bigcap C^{2m-d-1}(\mathbb{R}^d) : \ \Delta^m g = 0, \ \text{ on } \mathbb{R}^d \setminus \mathbb{Z}^d \right\}, \tag{1}$$

where $\Delta := \sum_{i=1}^d \partial^2/\partial x_i^2$ is the Laplace operator and $\Delta^m$ is defined iteratively by $\Delta^m g = \Delta(\Delta^{m-1} g)$. Recently (see [5]) they have been generalized by considering the fundamental solutions of more general differential operators $\prod_{j=1}^m (-\Delta + \kappa_j^2 I)$.

It is well-known (see e.g. [6]) that $V_{2m}$ contains $\Pi_{2m-1}$ (the space of polynomials defined on $\mathbb{R}^d$ of degree not exceeding $2m-1$), and that for any $n \leq 2m-1$ it is possible to construct quasi-interpolation (QI) operators reproducing $\Pi_n$ (see e.g. [7,8]). The generator $\phi \in V_{2m}$ of the QI operator

$$Q_\phi(x, h, f) = \sum_{l \in \mathbb{Z}^d} f(hl)\phi(x/h - l) \tag{2}$$

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is required to decay fast enough at infinity, so that $Q_\phi(x, h, f)$ is well defined for $f$ growing at infinity not faster than a polynomial of degree $n$. The QI operator (2) approximates smooth enough functions with $L^\infty$-error of order $h^{n+1}$ [7].

Important bases of $V_{2m}$ are $m$-harmonic B-splines, [9,4]. In the following we denote them by $\phi_0$. They decay sufficiently fast, at least as $\|x\|^{-d-2}$ when $\|x\| \to \infty$, they generate quasi-interpolation operators reproducing linear polynomials and then, for $f$ with bounded derivatives of order 1, 2, 3, the sum

$$Q_0(x, h, f) = \sum_{l\in\mathbb{Z}^d} f(hl)\phi_0(x/h - l)$$

approximates $f$ in $\mathbb{R}^d$ with $L^\infty$-error of order $h^2$. In addition they are valid scaling function for generating a multiresolution analysis of $L^2(\mathbb{R}^d)$ [4].

To obtain higher approximation orders, the function $\phi_0$ should be replaced by a function, with a stronger decay rate as $\|x\| \to \infty$. For this reason we have presented in [10] a procedure which starts from a simple generator $\phi_0$, and recursively defines generators $\phi_1, \phi_2, \ldots, \phi_{m-1}$ with corresponding QI operators reproducing $\Pi_3, \Pi_5, \ldots, \Pi_{2m-1}$ respectively.

The aim of this paper, is to study other important properties of the generators $\phi_j$ defined in [10]. We prove that these new generators are positive definite functions, are scaling functions whenever $\phi_0$ has those properties. Moreover $\phi_0$ and $\phi_j$ generate the same multiresolution analysis, and we show that it is possible to define a convergent subdivision scheme and to provide a computational refinement which allows us a fast computation of

$$Q_j(x, h, f) = \sum_{l\in\mathbb{Z}^d} f(hl)\phi_j(x/h - l).$$

The paper is organized as follows. In Section 2 we give some definitions, preliminary results, and we briefly recall the construction of the new generators $\phi_j$. In Section 3 we give the properties of $\phi_j$, and in Section 4 we show some numerical experiments.

## 2. Preliminaries

### 2.1. Definitions, notations, useful results

We use in this paper the multi-index notation. In particular for $k \in \mathbb{Z}^d$, $|k| = \sum_{i=1}^d |k_i|$. The symbol $*$ denotes the convolution operator between functions or sequences and the semi-discrete convolution between a sequence and a function. $T := [-\pi, \pi]^d$ is the $d$-dimensional torus, $L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$ is the set of functions locally square integrable and $2\pi\mathbb{Z}^d$-periodic. In the sequel we assume $2m > d$.

We say that a linear transformation $A$ on $\mathbb{R}^d$ is an *acceptable dilation* for $\mathbb{Z}^d$ if it leaves $\mathbb{Z}^d$ invariant, i.e. $A\mathbb{Z}^d \subset \mathbb{Z}^d$, and all the eigenvalues of $A$ satisfy $|\lambda_i| > 1$. These properties imply that $|\det A|$ is an integer $\geq 2$. Important acceptable dilations for $Z^d$ are *similarities*, i.e matrices of the form $A = \rho A_0$, where $A_0$ is an orthogonal matrix, and $\rho$ is a real number such that $|\det A| = |\rho^d|$ is an integer $\geq 2$. In the case $d = 2$ these matrices are all of the form

$$\begin{pmatrix} q & p \\ -p & q \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} q & p \\ p & -q \end{pmatrix},$$

with $q, p \in \mathbb{Z}$. They include the classic dyadic matrix $2I$ and the quincunx case $q = p = 1$ which gives $|\det A| = 2$.

A multiresolution analysis (MRA) of dilation $A \in \mathbb{R}^{d\times d}$ is a sequence $\mathcal{V} = \{\mathcal{V}_l, l \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ satisfying the following properties

(i) $\mathcal{V}_l \subset \mathcal{V}_{l+1}$,
(ii) $\bigcup_{l\in\mathbb{Z}} \mathcal{V}_l$ is dense in $L^2(\mathbb{R}^d)$ and $\bigcap_{l\in\mathbb{Z}} \mathcal{V}_l = \{0\}$,
(iii) $f(\cdot) \in \mathcal{V}_l \Leftrightarrow f(A^{-1}\cdot) \in \mathcal{V}_{l-1}$,
(iv) there is a function of $\mathcal{V}_0$, called the scaling function, whose integer translates form a Riesz basis of $\mathcal{V}_0$.

Condition (iv) is equivalent to saying that $\mathcal{V}_0$ is the closed linear span of $\mathbb{Z}^d$ translates of one function $\phi$, which satisfies the condition

$$c_1 \leq \sum_{k\in\mathbb{Z}^d} |\hat{\phi}(\omega - 2\pi k)|^2 \leq c_2$$

for some positive constants $c_1, c_2$. The whole multiresolution analysis may be regarded as being generated by $\phi$. Note that the scaling function is not unique (see e.g. [11]). It is useful to remember the following results stated in [11, Section 2.3].

**Lemma 1.** *Let $\phi \in L^1(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$ be such that $\hat{\phi}(0) = 1$ and $\hat{\phi}(2k\pi) = 0$ for all $k \in \mathbb{Z}^d \setminus \{0\}$, then $\phi$ is a valid scaling function for generating a MRA $\mathcal{V}$ associated with $(\mathbb{Z}^d, A)$, if and only if $\phi$ enjoys the following properties:*

(1) *there is a function $S(\omega) \in L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$ such that*

$$\hat{\phi}(\omega) = S(B^{-1}\omega)\hat{\phi}(B^{-1}\omega) \quad \text{where } B = A^T; \tag{3}$$

(2) *for almost all $\omega$*

$$\sum_{k\in\mathbb{Z}^d} |\hat{\phi}(\omega - 2\pi k)|^2 > 0. \tag{4}$$