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A meshless discrete Galerkin (MDG) method for the numerical solution of integral equations with logarithmic kernels

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ABSTRACT

This paper describes a computational method for solving Fredholm integral equations of the second kind with logarithmic kernels. The method is based on the discrete Galerkin method with the shape functions of the moving least squares (MLS) approximation constructed on scattered points as basis. The MLS methodology is an effective technique for the approximation of an unknown function that involves a locally weighted least square polynomial fitting. The numerical scheme developed in the current paper utilizes the non-uniform Gauss–Legendre quadrature rule for approximating logarithm-like singular integrals and so reduces the solution of the logarithmic integral equation to the solution of a linear system of algebraic equations. The proposed method is meshless, since it does not require any background mesh or domain elements. The error analysis of the method is provided. The scheme is also applied to a boundary integral equation which is a reformulation of a boundary value problem of Laplace's equation with linear Robin boundary conditions. Finally, numerical examples are included to show the validity and efficiency of the new technique.

1. Introduction

In this paper, we investigate a method for obtaining the numerical solution of the logarithmic Fredholm integral equation of the second kind, namely

$$u(x) - \lambda \int_a^b K(x, y)u(y)\mathrm{d}y = f(x), \quad a \le x \le b, \tag{1}$$

where the right hand side function $f(t)$ and the logarithmic kernel function $K(x, y)$ are given, $u(x)$ is the unknown function to be determined, λ is a constant. Assume that the kernel function $K(x, y)$ takes the form

$$K(x, y) = \sum_{k=1}^{n} a_k(x, y) \ln |g_k(x - y)| + b_k(x, y), \tag{2}$$

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where $a_k(x, y)$ and $b_k(x, y)$ are well-behaved functions (that is, they are several times continuously differentiable) and the continuous functions $g_k(x-y)$ are such that the kernel function $K(x, y)$ has the logarithm-like singularity along the diagonal $x = y$ but is continuous elsewhere.

An example of such equations is the reformulation of the exterior boundary value problem for the two-dimensional Helmholtz equation which is characterized by [1]

$$u(x) + \int_{-\pi}^{\pi} K(x, y)u(y)\mathrm{d}y = f(x), \quad -\pi \le x \le \pi, \tag{3}$$

where the kernel $K(x, y)$ is logarithmic given by

$$K(x, y) = -\frac{a(x, y)}{\pi} \ln \left| 2 \sin \frac{x-y}{2} \right| + b(x, y), \tag{4}$$

with

$$a(x, y) = a_0 + a_1(x, y) \sin^2 \frac{x-y}{2}, \tag{5}$$

where a_0 is a constant, $a_1(x, y)$ and $b(x, y)$ are continuous functions of (x, y) and 2π periodic in each variable.

The few topics where a formulation of a problem by means of integral equations with logarithmic kernels has been used are reported in [2,1] as follows:

1. Investigation of electrostatic, and low frequency electromagnetic problems [3].
2. Methods for computing the conformal mapping of a given domain [4].
3. Solution of electromagnetic scattering problems [5,6].
4. Determination of propagation of acoustical and elastical waves [7,8].

Fredholm integral equations with logarithmic kernels are usually difficult to solve analytically so, it is required to obtain an efficient approximate solution. The projection and discrete projection methods [9–11] are the commonly used approaches for the numerical solutions of these types of integral equations. The discrete Petrov–Galerkin methods [12], piecewise polynomial collocation and Galerkin methods [13,14], Sinc-collocation methods [15], hybrid collocation methods [16], high-order collocation methods [17], iterated fast multiscale Galerkin methods [18], Bubnov–Galerkin methods [19] and Galerkin-wavelet methods [20–22] have been applied to solve weakly singular Fredholm integral equations of the second kind, especially with logarithmic kernels. The Nystrom (quadrature) [23–25] and the product integration methods as the development of Nystrom methods have been used to solve logarithmic Fredholm integral equations in [10,26,27]. Khuri and Wazwaz [28] have investigated Adomian decomposition methods for solving logarithmic Fredholm integral equations. An integral equation whose kernel presents logarithmic singularity has been numerically solved by the method of arbitrary collocation points (ACP) [29].

In recent years much attention has been paid to the meshless methods not only by applied mathematicians but also in the engineering community. The meshless methods are based upon the scattered data approximations that estimate a function without any mesh generation on the domain. These methods come in various favors, most of which can be explained either by what is known in the literature as radial basis functions (RBFs) [30–33], or in terms of the moving least squares (MLS) method [34]. The MLS consists of a local weighted least square fitting, valid on a small neighborhood of a point and only based on the information provided by its N closet points. The main advantage of using the MLS approximation is that it sets up and solves many small systems, instead of a single, but large, system [34,35].

The MLS technique [36,37] has significant importance applications in different problems of the numerical mathematics such as partial differential equations (PDEs). Using this approach establishes some new meshfree methods for solving PDEs, for example, the element-free Galerkin (EFG) method [38], boundary node method (BNM) [39], hp-cloud method [40], meshless local boundary integral equation (LBIE) method [41–43], Galerkin boundary node method (GBNM) [44,45], meshless local Petrov–Galerkin (MLPG) method [46,47] and so on.

Here, we would like to review some of the most recent works which investigated the numerical solutions of integral equations using the meshless methods. The meshless discrete collocation schemes based on the MLS approximations have been used for the numerical solution of linear and nonlinear integral equations on non-rectangular domains in [48,49] and integro-differential equations in [50] with sufficiently smooth kernels. Authors of [44,51] have proposed a MLS-based meshless Galerkin method, the Galerkin boundary node method (GBNM), for boundary integral equations and also provided the error bound and the rate of convergence for this method. The radial basis functions (RBFs) [30,31,33] have been applied for solving linear and nonlinear two-dimensional integral equations on non-rectangular domains with the error analysis in [52,53]. Also we refer the interested reader to [36,37] for some recent investigation on MLS.

In this article, we employ the moving least square (MLS) approximation to solve the logarithmic Fredholm integral equation of the second kind (1). The scheme utilizes shape functions of the MLS approximation constructed on distributed nodal points on the interval $[a, b]$ to approximate the unknown function u in the discrete Galerkin method. We also apply the proposed approach to a boundary integral equation that arises from the boundary value problem for Laplace's equation

$$\begin{cases} \Delta u(\mathrm{x}) = 0, & \mathrm{x} \in D \subset \mathbb{R}^2, \\ \dfrac{\partial u(\mathrm{x})}{\partial n_{\mathrm{x}}} + p(\mathrm{x})u(\mathrm{x}) = g(\mathrm{x}), & \mathrm{x} \in \partial D, \end{cases} \tag{6}$$