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Algorithms for the Geronimus transformation for orthogonal polynomials on the unit circle ☆

Matthias Humet *, Marc Van Barel

Department of Computer Science, KU Leuven, Celestijnenlaan 200a - bus 2402, 3001 Heverlee, Belgium

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ABSTRACT

Let $\hat{\mathcal{L}}$ be a positive definite bilinear functional on the unit circle defined on $\mathbb{P}_n$, the space of polynomials of degree at most n. Then its Geronimus transformation $\mathcal{L}$ is defined by $\hat{\mathcal{L}}(p, q) = \mathcal{L}\big((z-\alpha)p(z), (z-\alpha)q(z)\big)$ for all $p, q \in \mathbb{P}_n, \alpha \in \mathbb{C}$. Given $\hat{\mathcal{L}}$, there are infinitely many such $\mathcal{L}$ which can be described by a complex free parameter. The Hessenberg matrix that appears in the recurrence relations for orthogonal polynomials on the unit circle is unitary, and can be factorized using its associated Schur parameters. Recent results show that the unitary Hessenberg matrices associated with $\mathcal{L}$ and $\hat{\mathcal{L}}$, respectively, are related by a QR step where all the matrices involved are of order $n + 1$. For the analogue on the real line of this so-called spectral transformation, the tridiagonal Jacobi matrices associated with the respective functionals are related by an LR step. In this paper we derive algorithms that compute the new Schur parameters after applying a Geronimus transformation. We present two forward algorithms and one backward algorithm. The QR step between unitary Hessenberg matrices plays a central role in the derivation of each of the algorithms, where the main idea is to do the inverse of a QR step. Making use of the special structure of unitary Hessenberg matrices, all the algorithms are efficient and need only $\mathcal{O}(n)$ flops. We present several numerical experiments to analyse the accuracy and to explain the behaviour of the algorithms.

1. Introduction

Let $\mathbb{L}_{m,n},\ m \leq n$, be the vector space of Laurent polynomials

$$p(z) = \sum_{k=m}^{n} a_k z^k, \quad a_k \in \mathbb{C}.$$

We denote by $\mathbb{P} = \mathbb{L}_{0,\infty}$ the vector space of polynomials with complex coefficients and by $\mathbb{P}_n = \mathbb{L}_{0,n}$ its subspace with polynomials of degree less than or equal to n.

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* Corresponding author. Tel.: +32 16 327563; fax: +32 16 327996.
E-mail addresses: matthias.humet@cs.kuleuven.be (M. Humet), marc.vanbarel@cs.kuleuven.be (M. Van Barel).

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Next we consider a bilinear functional $\mathcal{L}$ defined on $\mathbb{P}_n$, which is Hermitian, $\overline{\mathcal{L}(p,q)} = \mathcal{L}(q,p)$, and unitary, $\mathcal{L}(zp,zq) = \mathcal{L}(p,q)$. The moment matrix $T_n \in \mathbb{C}^{(n+1)\times(n+1)}$ associated with $\mathcal{L}$ is the Toeplitz matrix

$$T_n = \left[\mathcal{L}(z^i, z^j)\right]_{i,j=0}^n = \begin{pmatrix} \mu_0 & \overline{\mu_1} & \cdots & \overline{\mu_{n-1}} & \overline{\mu_n} \\ \mu_1 & \mu_0 & \ddots & & \overline{\mu_{n-1}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mu_{n-1} & & \ddots & \mu_0 & \overline{\mu_1} \\ \mu_n & \mu_{n-1} & \cdots & \mu_1 & \mu_0 \end{pmatrix} \tag{1}$$

where $\mu_k = \mathcal{L}(z^k, 1),\ k = 0, \ldots, n$, are the moments associated with $\mathcal{L}$.

Definition 1.1 ([1]).

(i) $\mathcal{L}$ is quasi-definite on $\mathbb{P}_n$ if T_n is strongly regular, i.e., if all the leading principal submatrices of T_n are nonsingular.
(ii) $\mathcal{L}$ is positive definite on $\mathbb{P}_n$ if $T_n > 0$, i.e., if T_n is positive definite.

For $\mathbb{P}_\infty = \mathbb{P}$, we will simply say that $\mathcal{L}$ is quasi-definite (or positive definite).

As mentioned in [2], $\mathcal{L}$ can be written as $\mathcal{L}(p,q) = \mathcal{F}\left(p(z)\overline{q}\left(\frac{1}{z}\right)\right)$, where $\mathcal{F}$ is a linear functional defined on $\mathbb{L}_{-n,n}$ with

$$\mathcal{F}(z^k) = \begin{cases} \mu_k & k \geq 0, \\ \overline{\mu_{-k}} & k \leq 0. \end{cases}$$

The bar on $\overline{q}(z)$ denotes complex conjugation of the coefficients of $q(z)$. It is well known that if $\mathcal{L}$ is positive definite, then it has an integral representation given by

$$\mathcal{L}(p,q) = \int_{\mathbb{T}} p(z)\overline{q}\left(\frac{1}{z}\right) d\mu(z), \tag{2}$$

where $d\mu(z)$ is a positive measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (see [2–4]).

If $\mathcal{L}$ is positive definite on $\mathbb{P}_n$, then there exists a unique sequence of orthonormal polynomials $\{\phi_k\}_0^n$ defined by[1]

$$\begin{aligned} &\phi_k(z) = \kappa_k z^k + \text{ lower degree terms}, \quad \kappa_k > 0, \\ &\mathcal{L}(\phi_k, \phi_l) = \delta_{k,l}, \end{aligned} \tag{3}$$

where $\delta_{k,l}$ is the Kronecker delta. The polynomials $\phi_k(z)$ satisfy the following recurrence relations (see [5])

$$\begin{aligned} &\rho_k\phi_k(z) = z\phi_{k-1}(z) + a_k\phi_{k-1}^*(z), \\ &\phi_k(z) = z\rho_k\phi_{k-1}(z) + a_k\phi_k^*(z), \end{aligned}$$

where $\rho_k = \sqrt{1-|a_k|^2}$ and $\phi_k^*(z) = z^k\,\overline{\phi_k}\,(1/z)$ is the so-called reversed polynomial of $\phi_k(z)$. Both recurrence relations are equivalent, the first is a forward relation, while the second is a backward one. The numbers $\{a_k\}_1^n$ are known as Schur parameters, Verblunsky parameters and reflection coefficients. They lie inside the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and together with the first moment μ_0, they determine the polynomials $\{\phi_k\}_0^n$ completely.

If $\mathcal{L}$ is quasi-definite but not positive definite on $\mathbb{P}_n$, then there exists a sequence of monic orthogonal polynomials $\{\Phi_k\}_0^n$, so $\Phi_k(z) = z^k +$ lower degree terms, and $\mathcal{L}(\Phi_k, \Phi_l) = \gamma_k\delta_{k,l}$ with $\gamma_k \neq 0$. The polynomials $\Phi_k(z)$ satisfy similar recurrence relations determined by the same Schur parameters $\{a_k\}_1^n$, which satisfy $|a_k| \neq 1$. Note that in the quasi-definite case, the Schur parameters can lie outside the unit circle while $a_k \in \mathbb{D}$ for all $k \leq n$ iff $\mathcal{L}$ is positive definite on $\mathbb{P}_n$.

Suppose $\mathcal{L}$ is positive definite, with associated orthonormal polynomials $\{\phi_k\}_0^\infty$ and $\boldsymbol{\phi}(z) = [\phi_0(z)\ \phi_1(z)\ \cdots]^T$, then

$$z\,\boldsymbol{\phi}(z)^T = \boldsymbol{\phi}(z)^T H \tag{4}$$

where H is a semi-infinite Hessenberg matrix with orthonormal columns (see [5, Chapter 4]). It is completely determined by the sequence of Schur parameters $\{a_n\}_1^\infty$ that appear in the recurrence relations for $\{\phi_k\}_0^\infty$. We call H the Hessenberg matrix associated with $\mathcal{L}$.

In [6,2,7,8] the following linear spectral transformation of $\mathcal{L}$ has been studied

$$\hat{\mathcal{L}}(p,q) = \mathcal{L}\big((z-\alpha)p(z), (z-\alpha)q(z)\big), \quad \alpha \in \mathbb{C}, \tag{5}$$

called the Christoffel transformation of $\mathcal{L}$. We use the notation $\hat{\mathcal{L}} = |z-\alpha|^2\mathcal{L}$, since for a positive definite $\mathcal{L}$ associated with a measure $d\mu(z)$, it amounts to multiplying the measure by $|z-\alpha|^2$. Our work builds on an important connection

[1] We will always assume that orthonormal polynomials are defined with a positive highest degree coefficient.