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Stability of analytical and numerical solutions of nonlinear stochastic delay differential equations☆

Siqing Gan [a,*], Aiguo Xiao [b], Desheng Wang [c]

[a] School of Mathematics and Statistics, State Key Laboratory of High Performance Complex Manufacturing, Central South University, Changsha, Hunan 410083, China

[b] School of Mathematics and Computational Science, Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Xiangtan, Hunan 411105, China

[c] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore, 637371, Singapore

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ABSTRACT

This paper concerns the stability of analytical and numerical solutions of nonlinear stochastic delay differential equations (SDDEs). We derive sufficient conditions for the stability, contractivity and asymptotic contractivity in mean square of the solutions for nonlinear SDDEs. The results provide a unified theoretical treatment for SDDEs with constant delay and variable delay (including bounded and unbounded variable delays). Then the stability, contractivity and asymptotic contractivity in mean square are investigated for the backward Euler method. It is shown that the backward Euler method preserves the properties of the underlying SDDEs. The main results obtained in this work are different from those of Razumikhin-type theorems. Indeed, our results hold without the necessity of constructing or finding an appropriate Lyapunov functional.

1. Introduction

Many physical, engineering and economic processes can be modeled by stochastic differential equations (SDEs). The rate of change of such a system depends only on its present state and some noisy input. However, in many practical situations the rate of change of the state depends not only on the present but also on the past states of the system. Stochastic functional differential equations (SFDEs) give a mathematical formulation for such system. For more details on SFDEs, we refer to [1–3] and the references therein.

SFDEs also can be regarded as a generalization of deterministic functional differential equations when stochastic effects are taken into account. For deterministic Volterra functional differential equations (VFDEs) in Banach spaces, Li [4] discussed the stability, contractivity and asymptotic stability of the solutions. In [4], the author introduced the so-called $\frac{1}{n}$-perturbed problem and constructed an auxiliary function $Q(t)$ for the corresponding study. The $\frac{1}{n}$-perturbed problem can be used to

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* Corresponding author. Tel.: +86 13667342362.
E-mail addresses: sqgan@csu.edu.cn, siqing_gan@yahoo.com (S. Gan).

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deal with a wide variety of delay arguments and the auxiliary function $Q(t)$ is the crux to establish the main results in [4]. The work [4] provides a unified framework for stability analysis of nonlinear stiff problems in ordinary differential equations, delay differential equations, integro-differential equations and VFDEs of other types. The theory in [4] was further extended to nonlinear Volterra neutral functional differential equations (VNFDEs) [5]. Moreover, in [6], it is proved that the implicit Euler method preserves the stability of VFDEs and VNFDEs.

It is natural to ask whether the solutions of SFDEs possess similar properties to those presented in [4] and which methods can reproduce the properties. Due to the unique features of stochastic calculus, the numerical analyses of SFDEs significantly differ from those developed for the numerical analyses of their deterministic counterparts. In the literature, much attention on numerical stability has been focused on a special class of SFDEs, namely, stochastic delay differential equations (SDDEs); see [7–12]. The results mainly concern the mean-square stability, asymptotic stability and exponential mean-square stability for SDDEs with bounded lags. Very recently, Fan, Song and Liu [13] discussed the mean-square stability of semi-implicit Euler methods for linear stochastic pantograph equations. Far less is known for long-run behavior of nonlinear SDDEs with unbounded lags. Moreover, to our best knowledge, there is no work on the contractivity analysis of numerical methods for SDDEs. Our aims in this paper are to investigate the stability and contractivity of nonlinear SDDEs with bounded and unbounded lags and to study the numerical preservation of those the properties. The main results of this paper could be summarized as follows.

(i) Sufficient conditions for the stability, contractivity and asymptotic contractivity in mean square of the solutions for nonlinear SDDEs are derived. The results provide a unified theoretical treatment for SDDEs with constant delay and variable delay (including bounded and unbounded variable delays). Applicability of the theory is illustrated by linear and nonlinear SDDEs with a wide variety of delay arguments such as constant delays, piecewise constant arguments, proportional delays and so on. The theorems established in this paper work for some SDDEs to which the existing theories cannot be applied. Our main results of analytic solutions can be regarded as a generalization of those in [4] restricted in finite-dimensional Hilbert spaces and finitely many delays to the stochastic version.

(ii) It is proved that the backward Euler method preserves the stability, contractivity and asymptotic contractivity in mean square of the underlying systems. In particular, Theorems 4.2 and 4.4 show that the backward Euler method preserves the contractivity and asymptotic contractivity without any constraint on the numerical stepsize.

We point out that the main theorems in the present paper are different from the Razumikhin-type theorems established in [7,1]. Our theorems can be directly applied to establish the stability without the necessity of constructing and finding an appropriate Lyapunov functional, as required by the Razumikhin-type theorems. In this sense, our theorems are more convenient for stability analysis than the Razumikhin-type theorems.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and assumptions, which will be used throughout the rest of the paper. In Section 3, some criteria for the stability, contractivity and asymptotic contractivity in mean square of solutions for nonlinear SDDEs are established. The main results obtained in this section are applied to SDDEs with bounded and unbounded lags, respectively. In Section 4, sufficient conditions for the stability, contractivity and asymptotic contractivity in mean square for the backward Euler method are derived. Stability of analytical and numerical solutions of SDDEs with several delays is discussed in Section 5.

2. Stochastic delay differential equations

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq a}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq a}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_a$ contains all the $\mathbb{P}$-null sets). Let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an m-dimensional Wiener process defined on the probability space. Let $\langle\cdot\rangle$ be inner product in $\mathbb{C}^d$ and $|\cdot|$ corresponding norm. In this paper, $|\cdot|$ also denotes the trace norm (F-norm) in $\mathbb{C}^{d\times m}$. Also, $C([t_1, t_2]; \mathbb{C}^d)$ is used to represent the family of continuous mappings ψ from $[t_1, t_2]$ to $\mathbb{C}^d$. Let $p > 2$ and denote by $L^p_{\mathcal{F}_t}([t_1, t_2]; \mathbb{C}^d)$ the family of $\mathcal{F}_t$-measurable $C([t_1, t_2]; \mathbb{C}^d)$-valued random variables $\psi = \{\psi(u) : t_1 \leq u \leq t_2\}$ such that $\|\psi\|^p_{\mathbb{E}} = \sup_{t_1\leq u\leq t_2} \mathbb{E}|\psi(u)|^p < \infty$. $\mathbb{E}$ denotes mathematical expectation with respect to $\mathbb{P}$.

Consider the following initial value problems of SDDEs in the sense of Itô

$$\begin{cases} dx(t) = f(t, x(t), x(t-\tau(t)))dt + g(t, x(t), x(t-\tau(t)))dw(t), \quad t \in [a, b], & \text{(a)} \\ x(t) = \xi(t), \quad t \in [a-\tau_0, a],\ \xi \in L^p_{\mathcal{F}_a}([a-\tau_0, a]; \mathbb{C}^d), & \text{(b)} \end{cases} \tag{2.1}$$

where a, b, τ_0 are constants with $-\infty < a < b < +\infty$ and $\tau_0 \geq 0, \tau(t) \geq 0, \inf_{a\leq t\leq b}(t-\tau(t)) \geq a-\tau_0$, $f : [a, b] \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d, g : [a, b] \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^{d\times m}$ are given continuous mappings. We assume that the drift coefficient f and the diffusion coefficient g satisfy the following conditions.

For each $R > 0$ there exists a constant C_R, depending only on R, such that

$$|f(t, x_1, y) - f(t, x_2, y)| \leq C_R|x_1 - x_2|, \quad |x_1| \vee |x_2| \vee |y| \leq R, \tag{2.2}$$

$$\Re\langle x_1 - x_2, f(t, x_1, y) - f(t, x_2, y)\rangle \leq \alpha(t)|x_1 - x_2|^2, \tag{2.3}$$

$$|f(t, x, y_1) - f(t, x, y_2)| \leq \beta(t)|y_1 - y_2|, \tag{2.4}$$

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq \gamma_1(t)|x_1 - x_2| + \gamma_2(t)|y_1 - y_2|, \tag{2.5}$$