Journal of Computational and Applied Mathematics 268 (2014) 34–55

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Preconditioned conjugate gradient method for finding minimal energy surfaces on Powell–Sabin triangulations

A.M. Sajo-Castelli [a,*], M.A. Fortes [b], M. Raydan [a]

[a] *Departamento de Cómputo Científico y Estadística, Universidad Simón Bolívar, 89000, Caracas, Venezuela*

[b] *Departamento de Matemática Aplicada, Universidad de Granada, C/ Severo Ochoa, 18071, Granada, Spain*

ARTICLE INFO

Article history:
Received 23 October 2012
Received in revised form 24 January 2014

Keywords:
Conjugate gradient method
Preconditioning
Powell–Sabin element
Minimal energy surfaces
Data approximation

ABSTRACT

We present an iterative proposal, based on the preconditioned conjugate gradient method, to solve the linear system associated to the problem of approximating a data set by a minimal energy surface constructed through a Powell–Sabin finite element over a Δ^1-type triangulation defined on a polygonal domain. These approximation problems give rise to symmetric, banded, and positive definite matrices with a very special block structure that depends on the basis functions of the associated vector space, and also on the numeration of the nodes in the triangulation. In practice, the associated sparse matrices are large and ill-conditioned. The special structure of these matrices allows us to adapt and explore several known preconditioned strategies to improve the performance of the conjugate gradient method.

We adapt and explore five different preconditioning strategies, including some well-known direct and also some recent inverse strategies. Special attention is paid to the delicate and difficult task of choosing the related parameters in each case. The quality of each preconditioner is evaluated by observing the clustering of the preconditioned matrix eigenvalues, and the obtained reduction in number of iterations. We report on seven different surfaces, and our results indicate that the best preconditioning strategies for this application are the ones based on incomplete factorizations. Nevertheless, from a computational-cost point of view, all the explored strategies are competitive.

1. Introduction

In this work we present and evaluate an iterative proposal, which overcomes the limitations of the direct methods currently used, to solve the linear system associated to the problem of approximating a data set by a minimal energy surface constructed through a Powell–Sabin finite element of class $\mathcal{C}^1$. Recent applications that make use of approximating surfaces in the presence of noisy data, or in the presence of data sets in which there are regions with absence of information (see e.g. [1–5]) underline the interest on this topic. Nevertheless, to our knowledge none of the papers in the existing literature regarding this field considers the highly important problem of how to solve the associated linear systems, which are large, very ill-conditioned, and possess a special block structure. The results presented in this paper, for a Powell–Sabin finite element of class $\mathcal{C}^1$, can be extended to other schemes to deal also with a large set of data points to be approximated by means of a surface and/or with higher dimensional spline spaces. These two aspects are of great interest in the field of engineering.

* Corresponding author. Tel.: +58 4129168599.
E-mail addresses: asajo@usb.ve, talassio@gmail.com (A.M. Sajo-Castelli), mafortes@ugr.es (M.A. Fortes), mraydan@usb.ve (M. Raydan).

http://dx.doi.org/10.1016/j.cam.2014.02.030

The matrix associated to the linear system, which is obtained when the related variational problem is discretized, has a very special block structure which depends on the basis functions chosen in the considered vector space, and on the numeration of the knots of the Δ^1-triangulation of the domain. Such matrix, called *stiffness matrix*, is always symmetric, banded, positive definite, and sparse, i.e., it has a very low percentage ($<1\%$) of nonzero entries. For high-dimensional vector spaces, the associated matrices are large and ill-conditioned. Therefore, in practice the use of iterative methods, such as the preconditioned conjugate gradient method, is highly recommendable. Usually, the choice of a suitable preconditioner for the conjugate gradient method is problem-dependent. For this special application, we adapt and explore five different preconditioning strategies, including some well-known direct and also some recent inverse strategies [6–9]. For every strategy a set of parameters must be chosen which have an important influence on the performance of the iterative method. We discuss the delicate and difficult task of choosing these parameters in each case. The quality of each preconditioner is evaluated by observing the clustering of the preconditioned matrix eigenvalues, and also by the obtained reduction in number of iterations. We present an extensive numerical report for the approximation of seven surfaces with different characteristics.

The rest of this paper is divided into sections as follows. In Section 2, we introduce the required notation, review the theoretical and practical aspects of the Powell–Sabin finite elements, and review their connection with the minimal energy approximating surface problem. In particular, in Section 2.1, we fully describe the sparse structure of the large-scale symmetric and positive definite stiffness matrix that appears when using Powell–Sabin finite elements of class $\mathcal{C}^1$. In Section 3, we briefly describe the preconditioned conjugate gradient method and its properties. We also present five different preconditioning strategies, and discuss their specialized adaptation to the linear systems described in Section 2. In Section 4, we present an extensive numerical report for the approximation of seven surfaces with different characteristics, using the specialized preconditioning strategies discussed in Section 3. Finally, in Section 5, we present some concluding remarks.

2. Powell–Sabin finite elements and minimal energy approximating surfaces

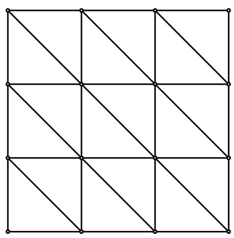
In general, a finite element [10–12] is a triplet

$$\left(\mathsf{K},\ \mathbb{E},\ \mathscr{P}\right),$$

where K is a geometric element in $\mathbb{R}^n$, $\mathbb{E}$ is a set of m data, usually related to the values of a given function and its derivatives at some points of K, and $\mathscr{P}$ is a vector space of dimension m, formed by polynomial functions defined over K in such a way that the following property holds: *for each fixed set of values of the data in* $\mathbb{E}$, *there exists a unique element in* $\mathscr{P}$ *taking such values.*

The main idea of the finite element method is to divide a polygonal domain $\mathbb{D}$ on which a problem (solving a partial differential equation or approximating a data set, among others) is stated into several elements $\{K_i\}$ in K, and then consider a functional vector space of functions over $\mathbb{D}$ formed by splines which are polynomials in $\mathscr{P}$ when restricted to each of the elements K_i. In particular, we work with polynomial splines over which we have the control of each data in $\mathbb{E}$ at each element K_i. In this work, we will consider triangular planar finite elements, i.e., we will consider K as the set of triangles, and $n = 2$. More precisely, we will work with the Powell–Sabin finite element of class $\mathcal{C}^1$, which is now described.

Introduction to the Powell–Sabin finite element of class $\mathcal{C}^1$. The Powell–Sabin finite element was initially developed to overcome the difficulties arising when dealing with contour plots of bivariate functions. The $\mathcal{C}^1$ Powell–Sabin finite element, introduced by M.J.D. Powell and M.A. Sabin in 1977 [13], has the advantage that the contour lines of the first partial derivatives of the functions are polygonal curves. In [13] the authors introduced also the more general $\mathcal{C}^k$ Powell–Sabin finite element. In order to describe the $\mathcal{C}^1$ Powell–Sabin finite element, we first need to introduce some basic concepts and notations: In the sequel, $\mathbb{D} \subset \mathbb{R}^2$ will denote a polygonal domain (open, polygonal, non-empty, bounded and connected set) such that $\overline{\mathbb{D}}$ admits a Δ^1-type triangulation, i.e., a triangulation induced by integer translates of $x = 0$, $y = 0$ and $x + y = 0$ (see e.g. [14]). The following figure shows an example of a Δ^1-triangulation of a square domain.



The Powell–Sabin sub-triangulation $\mathcal{T}_{\mathbf{PS}}$ associated to a triangulation $\mathcal{T}$ (see [15] or [13]) is obtained by joining the center O_T of the inscribed circle of each interior $T \in \mathcal{T}$ to the vertices of T and to the centers $O_{T'}$ of the inscribed circles of