Journal of Computational and Applied Mathematics 268 (2014) 56–67

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the quasilinear boundary-layer problem and its numerical solution

Relja Vulanović[a], Ljiljana Teofanov[b,*]

[a] Department of Mathematical Sciences, Kent State University at Stark, 6000 Frank Ave NW, North Canton, OH 44720, USA

[b] Department for Fundamental Disciplines, Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21000 Novi Sad, Serbia

ARTICLE INFO

Article history:
Received 13 August 2013
Received in revised form 20 February 2014

MSC:
65L10
65L12
65L20
65L70

Keywords:
Singular perturbation
Boundary-value problem
Shishkin mesh
Finite differences
Uniform convergence

ABSTRACT

We obtain improved derivative estimates for the solution of the quasilinear singularly perturbed boundary-value problem. This enables us to modify the transition point between the fine and coarse parts of the Shishkin discretization mesh. The resulting mesh may be denser in the layer than the standard Shishkin mesh. When this is the case, numerical experiments show an improvement in the accuracy of the computed solution.

1. Introduction

We consider the quasilinear singularly perturbed boundary-value problem,

$$Tu := -\varepsilon u'' - b(x,u)u' + c(x,u) = 0, \quad x \in I := [0,1], \qquad u(0) = u(1) = 0, \tag{1}$$

with $0 < \varepsilon \le \varepsilon^* \ll 1$ and functions b and c satisfying $b, c \in C^k(I \times \mathbb{R})$, $k \ge 1$, and

$$b(x,u) \ge \beta > 0, \qquad (b_x + c_u)(x,u) \ge 0, \quad x \in I, \ u \in \mathbb{R}. \tag{2}$$

The problem has a unique solution $u \in C^{k+2}(I)$, see [1]. The derivatives of u satisfy the following estimates [2]:

$$|u^{(i)}(x)| \le M \left(1 + \varepsilon^{-i} e^{-\beta x/\varepsilon}\right), \quad i = 0, 1, \dots, k+1, \ x \in I, \tag{3}$$

where M denotes a generic positive constant independent of ε. The estimates indicate that, in general, the solution u has a boundary layer of width $O(\varepsilon|\ln\varepsilon|)$ near $x = 0$. This is a region where u changes abruptly. All these properties are shared by the solution of the linear version of (1),

$$\begin{aligned} &\mathcal{L}u := -\varepsilon u'' - b(x)u' + c(x)u = f(x), \quad x \in I, \qquad u(0) = u(1), \\ &b, c, f \in C^k(I), \qquad b(x) \ge \beta > 0, \qquad c(x) \ge 0, \quad x \in I. \end{aligned} \tag{4}$$

* Corresponding author. Tel.: +381 21 4852278; fax: +381 21 6350770.
E-mail addresses: rvulanov@kent.edu (R. Vulanović), ljiljap@uns.ac.rs, ljiljap@eunet.rs (L. Teofanov).

http://dx.doi.org/10.1016/j.cam.2014.02.031
0377-0427/

The crucial result in this paper is an improvement of the estimates in (3). Specifically, we show that β can be replaced by β_0, where β_0 is a positive constant satisfying $b(0, u) \geq \beta_0$ in a closed, bounded domain. We get this result by using a refined analysis along the lines of [2]. There was no motivation to look for such improved derivative estimates in [2] because the numerical method proposed there does not use any specific constant β. However, special discretization meshes of Shishkin type have since then become one of the most popular tools for solving singular perturbation problems numerically, and a constant like β or β_0 is needed to construct the Shishkin mesh for the problem (1)–(2).

Numerical methods for singular perturbation problems have to be created carefully in order to achieve accuracy uniform in the perturbation parameter ε. One such method is to use an appropriate finite-difference scheme on the layer-adapted Shishkin mesh. Shishkin meshes are discussed at length in [3,4] and many numerical methods are known to be stable and accurate on it [5,6]. The Shishkin mesh is a piecewise-uniform mesh. It is divided into the fine part(s) which capture the layer(s) and the coarse part(s) outside the layer(s). The points at which the mesh step size changes are called transition points. The influence of the choice of the transition point and the complete mechanism of the Shishkin mesh are explained in detail in [7]. Improvements and generalizations of the Shishkin mesh can be found in [5,8].

The Shishkin mesh for (1)–(2) consists of a fine part near $x = 0$ and a coarse part outside the layer, with a transition point σ between these two parts. The standard definition of σ is $a\varepsilon \ln N/\beta$, where N is the total number of mesh steps and a is a sufficiently large positive parameter, the choice of which is related to the order of convergence of the numerical method.

The transition point $\sigma = a\varepsilon \ln N/b(0, 0)$ is used in [9] in numerical experiments with the quasilinear problem (1)–(2). It is reported there that the modified transition point produced more accurate numerical results than the standard transition point. However, the theoretical analysis in that paper is only done with the standard transition point because the error of the numerical method is analyzed using (3). This gave us the motivation to look for improved derivative estimates. We discovered that the asymptotic expansion of u in the general quasilinear case (1)–(2), [10, p. 135], does not seem to support the use of $b(0, 0)$ in the definition of σ, but rather indicates that $\sigma = a\varepsilon \ln N/\beta_0$ may be theoretically justified. This is what we analyze in the present paper.

The fact that $\beta_0 \geq \beta$ implies that the derivative estimates with β_0 are at least as sharp as those in (3). They are sharper if $\beta_0 > \beta$. No additional assumptions on the problem (1)–(2) are required for this result. For the numerical solution of the problem, $\beta_0 > \beta$ implies that the transition point of the Shishkin mesh with β_0 is closer to $x = 0$ than the standard one with β. Therefore, the modified mesh resolves the layer better and it is denser in the layer. Because of this, we can expect more accurate numerical results (but no change in the order of ε-uniform convergence). Our numerical experiments confirm this in a great majority of cases. Another practical advantage is that it may be easier to find β_0 than β. This is certainly so when the problem is linear, in which case β_0 reduces to $b(0)$. Moreover, the asymptotic expansion of the solution to the linear problem (4) has $\exp(-b(0)x/\varepsilon)$ for the boundary-layer function (see [10, p. 94] for instance). We can see from there that it is more natural to use $b(0)$ than β in the Shishkin mesh. Andreev and his collaborators, [11–13], come close to using the transition point with $b(0)$. Their transition point is of the form (using our notation) $C\varepsilon \ln N$ with $C > a/b(0)$. They allow for $C = a/b(0)$ only in the case of the homogeneous equation with constant coefficients.

In [14], we have already improved, in the above sense, the results from [11–13] for the linear problem. The present paper is a generalization of the [14] result to the quasilinear case. The generalization is not straightforward. The analysis in [14] is based on the solution representation due to Kellogg and Tsan [15], which cannot be extended to the quasilinear case. A similar Kellogg–Tsan representation of the solution is also used in [16] to achieve an analogous result for the singularly perturbed linear reaction–diffusion problem (with $b \equiv 0$ and a sufficiently smooth function c, such that $c > 0$ on I).

The outline of the paper is as follows. In Section 2 we discuss the continuous problem and derive the improved derivative estimates. The upwind finite-difference discretization of (1)–(2) on a general mesh is presented in Section 3. The modified Shishkin mesh is introduced in Section 4. In the same section we prove the almost-first-order pointwise ε-uniform convergence of the numerical solution and extend this result to its piecewise linear interpolant. Section 5 presents numerical experiments which confirm the theoretical results and show improvements of the computed solutions in comparison to those obtained on the standard Shishkin mesh.

2. The continuous problem

In this section we consider the boundary-value problem (1) satisfying (2). We assume that ε^* is sufficiently small. It should be mentioned that the condition $(b_x + c_u)(x, u) \geq 0$ in (2) is equivalent to saying that the function $b_x + c_u$ is bounded from below. If $(b_x + c_u)(x, u) \geq \gamma$, $x \in I$, $u \in \mathbb{R}$, for some $\gamma < 0$, then the transformation $u = \hat{u} \cdot \exp[2\gamma x/(\beta + \sqrt{\beta^2 + 4\gamma\varepsilon})]$ yields a problem in $\hat{u}$ which satisfies (2). Note that $\beta^2 + 4\gamma\varepsilon \geq 0$ with a sufficiently small ε^*.

For any function y, $y \in C(I)$, we define the norms

$$\|y\|_\infty = \max_{x \in I} |y(x)| \quad \text{and} \quad \|y\|_1 = \int_0^1 |y(x)|\, dx.$$

Recall that k indicates the smoothness of the functions b and c and that M stands for any (in the sense of $\mathcal{O}(1)$) positive constant which is independent of ε. Some particular constants of this kind are subscripted.