Journal of Computational and Applied Mathematics 268 (2014) 68–81

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Solving inverse Stokes problems by modified collocation Trefftz method

Chia-Ming Fan *, Hong-Huei Li, Chuan-Yen Hsu, Chun-Hung Lin

Department of Harbor and River Engineering & Computation and Simulation Center, National Taiwan Ocean University, Keelung 20224, Taiwan

ARTICLE INFO

Article history:
Received 24 October 2012
Received in revised form 11 February 2014

Keywords:
Bi-harmonic equation
Inverse Stokes problem
Modified collocation Trefftz method
Characteristic length
Meshless method

ABSTRACT

In this paper, the two-dimensional inverse Stokes problems, governed by bi-harmonic equations, are stably solved by the modified collocation Trefftz method (MCTM). In some practical applications of the Stokes problems, part of the boundary conditions cannot be measured in advance, so the mathematical descriptions of such problems are known as the inverse Stokes problems. When numerical simulation is adopted for solutions of the inverse Stokes problems, the solutions will become extremely unstable, which means that small perturbations in the boundary conditions will result in large errors of the final results. Hence, we adopted the MCTM for stably and efficiently analyzing the inverse Stokes problems. The MCTM is one kind of boundary-type meshless methods, so the mesh generation and the numerical quadrature can be avoided. Besides, the numerical solution is expressed as a linear combination of T-complete functions modified by a characteristic length. By enforcing the satisfactions of the boundary conditions at every boundary node, a system of linear algebraic equations will be yielded. The unknown coefficients in the solution expression can be acquired by directly inverting the coefficient matrix. The numerical solutions and their derivatives can be easily obtained by linear summation. Three numerical examples are provided to demonstrate the accuracy and the stability of the proposed meshless scheme for solving the two-dimensional inverse Stokes problems.

1. Introduction

The flow field of viscous incompressible fluid is governed by the well-known Navier–Stokes equations. When the inertial force of flow field is negligible, the two-dimensional Navier–Stokes equations will be reduced to the Stokes equations, which are used to describe the fluid motion for extremely viscous fluid or fluid with very slow velocity. In Stokes problem, the flow field is dominated by the pressure gradient force and the viscous force. In order to understand the underlying physics of Stokes flow, it is necessary to accurately analyze the Stokes equations. In comparing with mathematical analysis and experimental study, numerical simulation seems the better choice when realistic applications in irregular domains are considered. So, in this paper, the Stokes equations are converted to a bi-harmonic equation by utilizing the definition of streamfunction and a boundary-type meshless method, the modified collocation Trefftz method (MCTM), is used to accurately and efficiently analyze the two-dimensional inverse Stokes problems.

When numerical simulations are used for solutions of the Stokes equations, there are some different formulations that can be adopted, such as the primary-variables formulation [1,2], the velocity–vorticity formulation [3], and the streamfunction

* Corresponding author.
E-mail addresses: cmfan@ntou.edu.tw, fanchiaming@gmail.com (C.-M. Fan).

http://dx.doi.org/10.1016/j.cam.2014.02.029

formulation [4]. The unknown variables in the primary-variables formulation are velocity components and pressure. Since only the gradient terms of pressure appear in the momentum equations of the primary-variables formulation, it greatly increases the difficulty for numerical simulation. On the other hand, by introducing the vorticity vector, which is defined by taking curl to velocity vector, the velocity–vorticity formulation can be derived. Although the difficulty of calculations of pressure can be overcome, the number of unknowns is increased for three-dimensional problems, which will cost more computer power. In streamfunction formulation, the Stokes equations can be recast as one biharmonic equation for two-dimensional problem. The number of unknowns in streamfunction formulation is the minimum in comparing with other formulations, so it is very cheap and simple in the viewpoint of numerical simulation. Thus, the two-dimensional inverse Stokes problems governed by a bi-harmonic equation are considered in this study.

Recently, many studies are focused on developing numerical schemes for solving the direct and inverse Stokes problems. For example, Li and Li [5] proposed the penalty finite element method for Stokes problems with nonlinear slip boundary conditions, while Ye [6] adopted the least-squares finite element methods and domain decomposition technique to analyze the Stokes problems. For some realistic applications and engineering problems, parts of the boundary conditions cannot be directly measured and determined before numerical simulation. This kind of problems is known as the inverse problems, which are highly ill-conditioned and very unstable. Hence, it is very important to develop a stable and efficient numerical algorithm for solving inverse problems, especially for Stokes equations. Chen et al. [7] used the method of fundamental solutions (MFS) to obtain the numerical solutions of inverse Stokes problems, as Zeb et al. [8,9] adopted the boundary element method to analyze two-dimensional inverse Stokes problems. In this paper, we used a highly accurate meshless scheme for stably solving the two-dimensional inverse Stokes problems.

In order to avoid time-consuming mesh generation and numerical quadrature, many so-called meshless methods are proposed in the past decades, such as the meshless local Petrov–Galerkin method [10], the radial basis function collocation method [11–13], the local radial basis function collocation method [14], the element-free Galerkin method [15], the MFS [16–19], the MCTM [20–23], smoothed particle hydrodynamics [24], generalized finite difference method [25,26]. Among them, the MFS and the MCTM are two of the most promising boundary-type meshless methods. In MFS, the solution is expressed as a linear combination of fundamental solutions, which are located out of the computational domain. Although the numerical results by MFS is extremely accurate, the determination of the fictitious boundary for fundamental solutions is still a challenging problem and needed further investigations. On the other hand, the numerical solution in the MCTM is expressed as a linear combination of T-complete functions, which will be modified by a characteristic length. In previous studies, condition numbers of the coefficient matrices in the MCTM are greatly reduced by introducing the characteristic length [20–23,27]. In addition, the numerical solution by the MCTM is extremely accurate and stable even for the Cauchy problems [27], one kind of inverse problems. Since only boundary nodes are needed, the MCTM is very simple and will cost very little computational power. Therefore, we adopt the highly accurate MCTM to stably and efficiently analyze the inverse Stokes problem which is formulated by a bi-harmonic equation.

In this paper, a boundary-type meshless scheme is proposed to stably analyze the two-dimensional inverse Stokes problems. The primary-variables formulation of the Stokes equations will be reformulated to a bi-harmonic equation by introducing the vorticity and the streamfunction. Then, the highly accurate MCTM will be adopted to resolve this equation. The numerical solution of streamfunction is expressed as a linear combination of T-complete functions and the unknown coefficients will be obtained by enforcing the satisfactions of the boundary conditions at every boundary node. The stability of the proposed meshless scheme is validated by adding noise to the given boundary conditions. In Section 1, the motivation and the introduction of this study will be given. Then, the Stokes equations and the proposed numerical scheme will be described. Three numerical examples are provided to validate the stability and the efficacy of the proposed meshless method. Based on the numerical results and comparisons, some conclusions and discussions will be drawn.

2. Governing equations

When the inertial force is negligible in comparison with the viscous force and the pressure gradient force, the governing equations for fluid flow are the well-known two-dimensional Stokes equations, which are shown as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (x, y) \in \Omega, \tag{1}$$

$$-\frac{\partial p}{\partial x} + \nabla^2 u = 0, \quad (x, y) \in \Omega, \tag{2}$$

$$-\frac{\partial p}{\partial y} + \nabla^2 v = 0, \quad (x, y) \in \Omega, \tag{3}$$

where u and v are the x-directional and the y-directional velocity components, respectively. p is the pressure and ∇^2 is the Laplacian operator. Ω and $\partial\Omega = \Gamma \cup \gamma$ denote the computational domain and the whole boundary. The above system of equations can be solved with suitable boundary conditions for a well-posed problem. In the above system of partial differential equations, the velocity components and the pressure are coupled with each other, so it is not easy to analyze this system directly. In this paper, we would convert the Stokes equations to a bi-harmonic equation by introducing the streamfunction.