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On normal and skew-Hermitian splitting iteration methods for large sparse continuous Sylvester equations☆

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ABSTRACT

This paper is concerned with some generalizations of the Hermitian and skew-Hermitian splitting (HSS) iteration for solving continuous Sylvester equations. The main contents we will introduce are the normal and skew-Hermitian splitting (NSS) iteration methods for the continuous Sylvester equations. It is shown that the new schemes can outperform the standard HSS method in some situations. Theoretical analysis shows that the NSS methods converge unconditionally to the exact solution of the continuous Sylvester equations. Moreover, we derive the upper bound of the contraction factor of the NSS iterations. Numerical experiments further show the effectiveness of our new methods.

1. Introduction

Let m, n be positive integers. In this paper we consider the following class of linear matrix equation

$$AX + XB = F, \quad A \in C^{m\times m}, B \in C^{n\times n}, F \in R^{m\times n}, \tag{1.1}$$

where A and B are both large sparse and positive definite matrices, and $X \in C^{m\times n}$ is the solution matrix sought.

Let $\lambda(A)$ denote the spectrum of A and $\lambda(B)$ denote the spectrum of B. A necessary and sufficient condition for (1.1) to have a unique solution is that $\{\lambda|\lambda \in \lambda(A)\} \cap \{-\overline{\mu}|\mu \in \lambda(B)\} = \emptyset$ (see [1]), where $\overline{\mu}$ is the conjugate of μ. We assume that this condition is always satisfied in our discussion.

The continuous Sylvester equation (1.1) plays an important role in many fields and has numerous applications in the control and system theory [2–4], stability of linear systems [5], analysis of bilinear systems [6], power systems [7] and so on. Many authors have considered such a linear matrix equation problem, i.e., linear matrix equation (1.1) has been extensively studied. When the matrices A and B are dense, the Bartels–Stewart [8] and Golub–Nash–Van Loan algorithms [9] are attractive. In the former scheme the Schur factorizations of both A and B are computed; the latter scheme computes the Schur factorization of B and brings A into upper Hessenberg form by a unitary similarity transformation. When the matrices A and B are large and sparse, iterative methods such as the alternating-direction implicit (ADI) method may be attractive

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(see [10–13] and references there) where the rate of convergence depends on the choice of certain iteration parameters, the Smiths method [14], the block successive overrelaxation (BSOR) method [15], and the matrix sign function method are often the methods of choice for efficiently and accurately solving the continuous Sylvester equation (1.1).

The HSS iteration was first introduced by Bai, Golub and Ng in [16] for the solution of a broad class of non-Hermitian linear systems $Cx = b$. It was shown in [16] that the HSS iteration is unconditionally convergent when $\mathscr{H}(C) = \frac{1}{2}(C + C^*)$ is positive definite. And it was shown in the same paper that choosing a proper parameter can minimize an upper bound of the spectral radius of the iteration matrix associated with the stationary scheme. Due to its promising performance and elegant mathematical properties, the HSS scheme immediately attracted considerable attention, resulting in numerous papers devoted to various aspects of the new algorithm. In one direction, the method was extended to the solution of saddle point problems in [17–21]. Other significant developments include preconditioned variants of the HSS iteration [18,22–24], extension to certain singular systems [25], studies on the optimal selection of iteration parameters [18,19,26–29], and application of the HSS preconditioner to specific problems like convection–diffusion equations [30–32] and linear matrix equations [33,34], often with excellent results. In [33], Bai presented the HSS iteration method for solving large sparse continuous Sylvester equations with non-Hermitian and positive definite/semidefinite matrices. The unconditional convergence of the HSS iteration method is proved and an upper bound on the convergence rate is derived. Numerical results in the paper show that the HSS iteration method is efficient and robust solver for the continuous Sylvester equations.

In this paper, we consider some approaches to accelerate the HSS iteration. In actual applications, there are situations that a matrix may be more naturally split into its normal and skew-Hermitian parts (NS) rather than its Hermitian and skew-Hermitian parts (HS) due to the concrete structure and property of the coefficient matrix of the linear system (1.1) inherited from the original physical or mathematical problem. The circulant-plus-diagonal matrix of the form $C + \mathrm{i} * D$ is exactly such an example, which often arises from the image processing and the numerical solution of the Helmholtz equation, etc., where i is the imaginary unit.

The paper is organized as follows. In Section 2, we establish the NSS iteration method for the continuous Sylvester equation (1.1). In Section 3, we analyze the convergence property of the NSS iteration method. In Section 4, some numerical experiments are presented to illustrate the theoretical results and show the effectiveness of the NSS iteration method for solving the continuous Sylvester equations. Finally, some conclusions are given in Section 5.

The following notations will be used throughout the rest part of this paper. For $\otimes$ stands for the Kronecker product, I is the identity matrix, $\|\cdot\|_2$ denote the spectral norm and $\|\cdot\|_F$ denote the F norm. $A^\top$ on behalf of the transpose of the matrix A, and $\bar{a}$ is the conjugate of the constant a. In addition, O is the 0-matrix.

2. The NSS iteration method

In this section, we establish the NSS iteration method which is a generalized version of the HSS algorithm that can accelerate the HSS iteration method for solving the continuous Sylvester equations.

As is known to all that an arbitrary non-Hermitian matrix can be divided into the sum of its normal part and its skew-Hermitian part (see [5]), that is, we can generalize the HS splitting to the normal and skew-Hermitian (NS) splitting. We refer the readers to [35,36] for detailed discussion about normal matrices. So for the matrices A and B of Eq. (1.1), we have

$$A = \mathscr{N}(A) + \mathscr{S}(A) \quad \text{and} \quad B = \mathscr{N}(B) + \mathscr{S}(B),$$

where $\mathscr{N}(A)$ and $\mathscr{N}(B)$ are the normal parts of matrices A and B, respectively; $\mathscr{S}(A)$ and $\mathscr{S}(B)$ are the skew-Hermitian parts of matrices A and B, respectively. Let α and β be two given positive constants and I the identity matrix of suitable dimension. Then we have the following results:

$$\begin{aligned} A &= (\alpha I + \mathscr{N}(A)) + (\mathscr{S}(A) - \alpha I) \\ &= (\alpha I + \mathscr{S}(A)) + (\mathscr{N}(A) - \alpha I) \end{aligned}$$

and

$$\begin{aligned} B &= (\beta I + \mathscr{N}(B)) + (\mathscr{S}(B) - \beta I) \\ &= (\beta I + \mathscr{S}(B)) + (\mathscr{N}(B) - \beta I). \end{aligned}$$

So based on the above NS splittings of matrices A and B, now we can establish the following NSS iteration method for solving the continuous Sylvester equation (1.1) in an analogous fashion to the HSS iteration scheme. More precisely, we have the following algorithmic description of the NSS iteration method.

The NSS iteration method. Given an initial matrix $X^{(0)} \in C^{m\times n}$, for $k = 0, 1, 2, \ldots,$ until the iteration sequence $\{X^{(k)}\}$ converges to the exact solution of the continuous Sylvester equation (1.1), compute

$$\begin{cases} (\alpha I + \mathscr{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta I + \mathscr{N}(B)) = (\alpha I - \mathscr{S}(A))X^{(k)} + X^{(k)}(\beta I - \mathscr{S}(B)) + F, \\ (\alpha I + \mathscr{S}(A))X^{(k+1)} + X^{(k+1)}(\beta I + \mathscr{S}(B)) = (\alpha I - \mathscr{N}(A))X^{(k+\frac{1}{2})} + X^{(k+\frac{1}{2})}(\beta I - \mathscr{N}(B)) + F, \end{cases}$$

where α and β are two given positive constants.

Evidently, just like the HSS iteration methods, each iterate of the NSS iteration alternates between the matrices $\alpha I + \mathscr{N}(A)$, $\beta I + \mathscr{N}(B)$ and the matrices $\alpha I + \mathscr{S}(A)$, $\beta I + \mathscr{S}(B)$, analogously to the classical ADI iteration for partial differential