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Exponential polar factorization of the fundamental matrix of linear differential systems

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ABSTRACT

We propose a new constructive procedure to factorize the fundamental real matrix of a linear system of differential equations as the product of the exponentials of a symmetric and a skew-symmetric matrix. Both matrices are explicitly constructed as series whose terms are computed recursively. The procedure is shown to converge for sufficiently small times. In this way, explicit exponential representations for the factors in the analytic polar decomposition are found. An additional advantage of the algorithm proposed here is that, if the exact solution evolves in a certain Lie group, then it provides approximations that also belong to the same Lie group, thus preserving important qualitative properties.

1. Introduction

Given the non-autonomous system of linear ordinary differential equations

$$\frac{dU}{dt} = A(t)\,U, \quad U(0) = I \tag{1}$$

with $A(t)$ being a real analytic $N \times N$ matrix, the Magnus expansion allows one to represent the fundamental matrix $U(t)$ locally as

$$U(t) = \exp(\Omega(t)), \quad \Omega(0) = O, \tag{2}$$

where the exponent $\Omega(t)$ is given by an infinite series

$$\Omega(t) = \sum_{m=1}^{\infty} \Omega_m(t) \tag{3}$$

whose terms are linear combinations of integrals and nested commutators involving the matrix A at different times [1]. The series converges in the interval $t \in [0, \tau)$ such that $\int_0^\tau \|A(s)\|ds < \pi$ and the sum $\Omega(t)$ verifies $\exp \Omega(t) = U(t)$. Different approximations to the solution of (1) are obtained when the series of Ω is truncated, all of them preserving important qualitative properties of the exact solution. Magnus expansion has been widely used as an analytic tool in many different areas of physics and chemistry, and also numerical integrators have been constructed, which have proved to be highly competitive with other, more conventional numerical schemes in terms of accuracy and computational cost (see [2] and references therein).

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Although the representation (2)–(3) and the approximations obtained when the series is truncated have several advantages, it is not always able to reproduce all the qualitative features of $U(t)$. In particular, suppose that the matrix-valued function $A(t)$ is periodic with period T. Then the Floquet theorems ensure the factorization of the solution as a periodic part and a purely exponential factor: $U(t) = P(t)\,\exp(tF)$, where F and P are $N \times N$ matrices, $P(t+T) = P(t)$ for all t and F is constant. It is clear, then, that the Magnus expansion does not explicitly provide this structure. In that case, however, it is possible to reformulate the procedure so that both matrices $P(t)$ and F can be constructed recursively [3].

Another example concerns symplectic matrices. As is well known, the most general $2N \times 2N$ symplectic matrix M can be written as the product of two exponentials of elements in the symplectic group Lie algebra as $M = \exp(X)\,\exp(Y)$, and each of the elements is of a special type, namely $X = JS^a$ and $Y(t) = JS^c$, where J is the standard canonical matrix,

$$J = \begin{pmatrix} O_N & I_N \\ -I_N & O_N \end{pmatrix},$$

S^a is a real symmetric matrix that anti-commutes with J and S^c is a real symmetric matrix that commutes with J [4]. Since X is symmetric and Y is skew-symmetric, notice that the factorization $\exp(X)\,\exp(Y)$ is a special type of polar decomposition for matrix M. According with this property, if $A(t)$ belongs to the symplectic Lie algebra $\mathfrak{sp}(2N)$, i.e., it verifies $A^T J + JA = O$, then the fundamental solution $U(t)$ evolves in the symplectic group $\mathrm{Sp}(2N)$ (i.e., $U^T(t) J\, U(t) = J$ for all t) and therefore admits a factorization

$$U(t) = \exp(X(t))\,\exp(Y(t)), \tag{4}$$

where $X(t) = JS^a(t)$ is a symmetric matrix and $Y(t) = JS^c(t)$ is skew-symmetric. A natural question is whether the Magnus expansion can be adapted to treat this problem in order to provide explicit analytic expressions for both $X(t)$ and $Y(t)$, just as in the case of a periodic matrix $A(t)$. More generally, one might try to adapt the Magnus expansion to construct explicitly a polar factorization of form (4) for the fundamental matrix of (1). In other words, the idea is then to build explicitly the solution of (1) as (4) with both matrices $X(t)$ and $Y(t)$ constructed as series of form

$$X(t) = \sum_{i \ge 1} X_i(t), \qquad Y(t) = \sum_{i \ge 1} Y_i(t). \tag{5}$$

This issue is addressed in the sequel. More specifically, we present a procedure that allows us to compute recursively X_i, Y_i in terms of nested integrals and nested commutators involving the matrix $A(t)$. Moreover, these series are shown to be convergent, at least for sufficiently small times. Thus, in the convergence domain, we have explicit exponential representations for the factors $H(t)$ and $Q(t)$ in the analytic polar decomposition of $U(t)$:

$$U(t) = H(t)\,Q(t). \tag{6}$$

As is well known, given an arbitrary time-varying nonsingular real analytic matrix function $U(t)$ on an interval $[a, b]$ so that $U(t)$, $\dot{U}(t)$ and $U(t)^{-1}$ are bounded, there exists an analytic polar decomposition (6), with $Q(t)$ orthogonal, $H(t)$ symmetric (but not definite) and both are real analytic on $[a, b]$ [5].

The polar decomposition of time-varying matrices has proved to be useful in several contexts. Thus, for instance, it appears in numerical methods for computing analytic singular value decompositions [5] and as a path to inversion of time-dependent nonsingular square matrices [6]. Polar decomposition is also used in computer graphics and in the study of stress and strain in continuous media [7]. Since both factors possess best approximation properties, it can be applied in optimal orthogonalization problems [8].

Theoretical general results on decompositions of a time varying matrix $U(t)$ of class $\mathcal{C}^k$ can be found in [9], where sufficient conditions for the existence of QR, Schur, SVD and polar factorizations are given and differential equations for the factors are derived.

The procedure we present here for computing the factorization (4) for the fundamental matrix of (1) has several additional advantages. First, even when the series are truncated, the structure of the polar decomposition still remains for the resulting approximations. Second, the algorithm can be easily implemented in a symbolic algebra package and may be extended without difficulty to get convergent approximations to the analytic polar decomposition of a more general class of nonsingular time-dependent matrices and also of the exponential of constant matrices. Third, if $A(t)$ belongs to a certain matrix Lie subalgebra, so that $U(t)$ evolves in the corresponding Lie group, it provides approximations in this Lie group, and thus they preserve important qualitative features of the exact solution. The symplectic case considered before is a case in point here. Fourth, if $A(t)$ depends on some parameters, this procedure leads to approximate factorizations to the exact solution involving directly these parameters, which in turn allows one to analyze different regions of the parameter space with just one calculation. In this sense, it differs from other more numerically oriented techniques for computing the polar decomposition existing in the literature (e.g. [10,6,5]).

It is important to stress that the formalism proposed here is not specifically designed to get efficient numerical algorithms for computing the analytic polar decomposition of an arbitrary matrix $U(t)$, but instead it is oriented to get a factorization of form (4) for the fundamental matrix of Eq. (1) which could be specially well adapted when the coefficient matrix $A(t)$ involves one or more parameters and belongs to some special matrix Lie subgroup.