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On the numerical solution of two-phase Stefan problems with heat-flux boundary conditions

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ABSTRACT

A recently derived numerical algorithm for one-dimensional one-phase Stefan problems is extended for the purpose of two-phase moving boundary problems in which the second phase first appears only after a finite delay time; this can occur if the phase change is caused by a heat-flux boundary condition. In tandem with the Keller box finite-difference scheme, the so-called boundary immobilization method is used. An important component of the work is the use of variable transformations that must be built into the numerical algorithm to resolve the boundary-condition discontinuity that is associated with the onset of phase change. This allows the delay time until solidification begins to be determined, and gives second-order accuracy in both time and space.

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1. Introduction

Phase-change, or Stefan, problems in which a material melts or solidifies occur in a wide variety of natural and industrial processes. Mathematically, these are special cases of moving-boundary problems, in which the location of the front between the solid and liquid is not known beforehand, but must be determined as part of the solution. Although most activity has been devoted to the classical one-dimensional Stefan problem in which phase change occurs due to a constant heating or cooling temperature, of more practical relevance is the situation where a heat flux or convective boundary condition is imposed. In general, there are no analytical solutions available for these cases and thus numerical methods are necessary. Furthermore, of all the numerical methods that have been applied to the classical problem, only the heat balance integral method has been extended to the non-classical Stefan problem [1–6]. Whilst this method is popular, an unsatisfactory feature is the use of assumptions on the form of the solution that may not be generally valid; furthermore, the solution obtained is of indeterminate order of accuracy. Amongst the other available numerical methods, the boundary immobilization method, implemented in tandem with the Keller Box scheme, has recently emerged as the most accurate alternative for the classical problem [7], and the purpose of this paper is to demonstrate the use of this method for the non-classical problem.

There are, however, several obstacles which make the extension non-trivial. Unlike the problems analysed in [7], which considered a material that was initially at its melting temperature, phase change will not commence instantaneously if the material is not at this temperature, and a central part of the problem is to determine when this actually occurs. Thereafter, it is necessary to determine the location of the moving phase-change boundary and, as in [7], to consider how numerically to handle the new phase, given that its thickness is initially zero. An initial analysis of the problem was undertaken in [8], and a principal finding was that the phase-change front starts to grow as $(t - t_m)^{3/2}$, where t_m is the time taken for phase change to begin; although similar to that for ablation problems [9], the initial evolution of the front is different to that for the classical Stefan problem, which has behaviour $(t - t_m)^{1/2}$, with $t_m = 0$. However, the analysis in [8] was incomplete in several respects:

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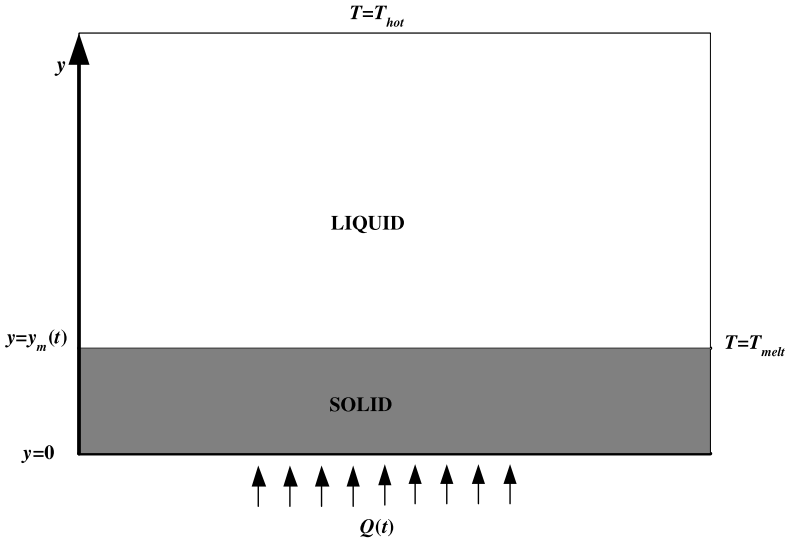

Fig. 1. System geometry.

1. the numerical scheme was not able to compute when phase change began;
2. no numerical scheme was given for the entire problem, only for the phase-change stage, and even this was only possible because an analytical expression was available for the initial condition for this stage;
3. the formal accuracy of the numerical scheme was not verified.

The layout of the paper is as follows. In Section 2, we formulate a problem for the solidification, due to a cooling heat flux, of material that is initially above its melting temperature; in Section 3, we nondimensionalize the model equations and transform them to a form more suitable for numerical integration. In Section 4, we explain how the Keller box scheme, in tandem with the boundary immobilization method, is applied to this particular problem. The results are then presented and discussed in Section 5, and conclusions are drawn in Section 6.

2. Mathematical formulation

Consider the cooling of a liquid, occupying the half-plane $y > 0$, that is initially at a temperature, T_{hot}, which is greater than its melting temperature, T_{melt}, and which is cooled at $y = 0$ for time $t > 0$ by an applied heat flux $Q(t)$. After cooling commences, the temperature of the liquid decreases until some time t_m, at which stage solid begins to form at $y = 0$; subsequently, solid occupies the region $0 \leq y \leq y_m(t)$ and liquid occupies $y > y_m(t)$, where $y_m(t)$ denotes the location of the solidification front. A schematic is shown in Fig. 1.

Assuming the material properties of the solid and liquid phases to be constant, the governing equations are as follows. For $0 < t < t_m$ and $y > 0$, and then $t > t_m$ and $y > y_m(t)$, we have

$$\rho_l c_{pl} \frac{\partial T_l}{\partial t} = k_l \frac{\partial^2 T_l}{\partial y^2}, \tag{1}$$

where T_l is the liquid temperature, k_l is the thermal conductivity of the liquid, c_{pl} is its specific heat capacity and ρ_l its density. For $t > t_m$ and $0 < y < y_m(t)$, we have

$$\rho_s c_{ps} \frac{\partial T_s}{\partial t} = k_s \frac{\partial^2 T_s}{\partial y^2}, \tag{2}$$

where T_s is the solid temperature, k_s is the thermal conductivity of the solid, c_{ps} is its specific heat capacity and ρ_s its density. We assume henceforth that $\rho_l = \rho_s = \rho$.

For boundary conditions, we have, at $y = y_m(t)$,

$$T_s = T_l = T_{melt}, \tag{3}$$

and the Stefan condition,

$$k_s \frac{\partial T_s}{\partial y} - k_l \frac{\partial T_l}{\partial y} = \rho \Delta H_f \frac{\mathrm{d}y_m}{\mathrm{d}t}, \tag{4}$$

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