Journal of Computational and Applied Mathematics 264 (2014) 71–81

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On evaluation of Bessel transforms with oscillatory and algebraic singular integrands ☆

Ruyun Chen [a,*], Congpei An [b]

[a] College of Science, Guangdong Ocean University, Zhanjiang, Guangdong 524088, PR China

[b] Department of Mathematics, Jinan University, Guangzhou, Guangdong 510632, PR China

ARTICLE INFO

Article history:
Received 25 July 2013
Received in revised form 7 January 2014

MSC:
65D32
65D30

Keywords:
Bessel function
Oscillatory integrals
Singular integrals
Taylor polynomial
Filon-type method

ABSTRACT

In this paper, we study efficient methods for computing the integrals of the form $\int_0^1 x^a(1-x)^b f(x)J_v(\omega x)dx$, where a, b, v, ω are the given constants and $\omega \gg 1$, J_v is the Bessel function of the first kind and of order v, f is a sufficiently smooth function on [0, 1]. Firstly, we express the moments in a closed form with the aid of special functions. Secondly, we induce the Filon-type method based on the Taylor interpolation polynomial at two endpoints and the Hermite interpolation polynomial at Clenshaw–Curtis points on evaluating the highly oscillatory Bessel integrals with algebraic singularity. Theoretical results and numerical experiments perform that the methods are very efficient in obtaining very high precision approximations if ω is sufficiently large.

1. Introduction

The problem of evaluating the highly oscillatory Bessel integrals

$$\int_0^1 x^a(1-x)^b f(x)J_v(\omega x)dx, \quad a > -1, b > -1, a+v > -1, \tag{1.1}$$

where f is a sufficiently smooth real-valued function, $J_v(\omega x)$ denotes the Bessel function of the first kind and of order v, and ω is large. It is well known that (1.1) has wide applications in physics and engineering sciences (see [1–5]). For $-1 < a, b < 0$ and $\omega \gg 1$, the integrand $x^a(1-x)^b f(x)J_v(\omega x)$ becomes highly oscillatory and singular, and presents serious difficulties in obtaining numerical convergence of the integrations. This means that some general numerical methods may not be immediately applicable to the integrand. It is easy to see that the study on (1.1) covers the more general integration interval $[0, \rho]$ by the following identity

$$\int_0^\rho x^a(\rho-x)^b f(x)J_v(\omega x)dx = \rho^{a+b+1}\int_0^1 t^a(1-t)^b g(x)J_v(\tilde{\omega} t)dt, \tag{1.2}$$

where $\tilde{\omega} = \rho\omega$ and $g(t) = f(\rho t)$.

When the frequency parameter ω in (1.1) is large, the integrand is highly oscillatory. In this case, a prohibitively large number of quadrature nodes is needed if one uses a quadrature rule based on polynomial interpolation of the integrand. It

☆ The work is supported partly by NSF of China (N0. 11071260, 11301222), NSF of Guangdong (No. S2012040007860), and NSF of Guangdong Ocean University (No. C13443).

* Corresponding author. Tel.: +86 7592383194.
E-mail addresses: csuchenruyun@yahoo.com.cn (R. Chen), andbach@163.com (C. An).

0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2014.01.009

implies that the use of standard quadrature rules for $\int_0^1 f(x)J_v(\omega x)dx$ is not efficient owing to the highly oscillatory character of the integrand (see [6–18]). Therefore, one has to resort to distinctive numerical methods. For $\int_0^1 f(x)J_v(\omega x)dx$, some special methods are used by integrating between zeros (see [6], pp. 118; [16,17]) or the modified Clenshaw–Curtis method is proposed [18]. Numerical approximation of $\int_a^b f(x)J_v(\omega x)dx$ (here $a \geq 0$ and $b > 0$) has been studied extensively in recent years (see [14–25]) and the references therein). To our knowledge, there is not so rich literature for the case of the oscillatory and singular integrands.

In 1928, Filon [12] introduced *Filon method* for oscillatory integrals of the forms

$$\int_a^b f(x)\sin(\omega x)dx \quad \text{and} \quad \int_0^\infty \frac{f(x)}{x}\sin(\omega x)dx. \tag{1.3}$$

For more details, one can refer to [12].

Based on the generalization of Filon's ideas, Iserles and Nørsett [13] developed *Filon-type method* for computing $\int_a^b f(x)e^{i\omega g(x)}dx$ where they were able to get better values by the use of the exact interpolation of f and its derivatives at the endpoints and stationary points. This can be accomplished using Hermite interpolation. Choose the points c_l and integers θ_l, $l = 1, \ldots, v$, and assume that there exists the polynomial $\tilde{f}$ which satisfies

$$\tilde{f}^{(j)}(c_l) = f^{(j)}(c_l), \quad j = 0, \ldots, \theta_l, l = 1, \ldots, v. \tag{1.4}$$

This interpolating polynomial can be written as a linear combination of function values and derivatives of f at the nodes c_l, that is,

$$\tilde{f}(x) = \sum_{l=1}^{v}\sum_{j=0}^{\theta_l} f^{(j)}(c_l)\psi_{l,j}(x). \tag{1.5}$$

Then the Filon-type method can be defined by

$$Q_s^F[f] = \int_a^b \tilde{f}(x)e^{i\omega g(x)}dx = \sum_{l=1}^{v}\sum_{j=0}^{\theta_l} w_{l,j}f^{(j)}(c_l), \tag{1.6}$$

where the general moments $w_{l,j} = \int_a^b \psi_{l,j}(x)e^{i\omega g(x)}dx$.

Clenshaw and Curtis [7] proposed a method for numerical integration that expands the integrand $f(x)$ in terms of Chebyshev polynomials in 1961. To avoid the Runge phenomenon, the use of Clenshaw–Curtis points is a much better choice than equally spaced points (see [9–11]). Inspired by this idea, the Clenshaw–Curtis method and the Clenshaw–Curtis–Filon-type method have been developed for computing $\int_a^b f(x)J_v(\omega x)dx$ (see [18,20,22]). In [20], Xiang, Cho, Wang and Brunner show the Clenshaw–Curtis points are not only a set of very good nodes for polynomial interpolations but also suitable for the computation of highly oscillatory integrals.

In this paper, we focus on the Filon-type method. With aid of two-point Taylor polynomial, we have a new scheme of the Filon-type method. Moreover, we draw our attention on the Clenshaw–Curtis points. As a result, we induce a Filon-type method at the Clenshaw–Curtis points for Bessel integrals with algebraic singular and oscillatory integrands.

In the sequel, we give the explicit computation of the moments $\int_0^1 x^a(1-x)^b J_v(\omega x)dx$ by means of some special functions. Section 2.2 shows the Filon-type method by means of two-point Taylor polynomial and the error of the proposed method. In Section 2.3 we give the Filon-type method based on the Clenshaw–Curtis points for (1.1) and show the uniform convergence of this method. Preliminary numerical examples show that two Filon-type method are efficient and accurate for approximating the integrals (1.1).

2. Numerical schemes

2.1. Evaluations of the generalized moments

In this section, we first show a formula for computing the *generalized moments*

$$\int_0^1 x^\alpha(1-x)^\beta J_v(\omega x)dx \tag{2.1}$$

in explicitly. Next, an order of (2.1) is given by means of the bounds of special functions.

The moments (2.1) can be denoted by the hypergeometric function (see [26], pp. 193)

$$\int_0^1 x^\alpha(1-x)^\beta J_v(\omega x)dx = \frac{\Gamma(\beta+1)\Gamma(\alpha+v+1)\,{}_2F_3\left(\frac{\alpha+v+1}{2}, \frac{\alpha+v+2}{2}; v+1, \frac{\alpha+\beta+v+2}{2}, \frac{\alpha+\beta+v+3}{2}; -\frac{\omega^2}{4}\right)}{2^v\omega^{-v}\Gamma(v+1)\Gamma(\alpha+\beta+v+2)},$$

$$\text{when } \Re(\beta) > -1, \Re(\alpha+\beta) > -2, \tag{2.2}$$