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Determination of a source in the heat equation from integral observations

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ABSTRACT

A novel inverse problem which consists of the simultaneous determination of a source together with the temperature in the heat equation from integral observations is investigated. These integral observations are weighted averages of the temperature over the space domain and over the time interval. The heat source is sought in the form of a sum of two space- and time-dependent unknown components in order to ensure the uniqueness of a solution. The local existence and uniqueness of the solution in classical Hölder spaces are proved. The inverse problem is linear, but it is ill-posed because small errors in the input integral observations cause large errors in the output source. For a stable reconstruction a variational least-squares method with or without penalization is employed. The gradient of the functional which is minimized is calculated explicitly and the conjugate gradient method is applied. Numerical results obtained for several benchmark test examples show accurate and stable numerical reconstructions of the heat source.

1. Introduction

Mathematical models related to inverse source problems arise in various practical settings, e.g. the determination of sources of water and air pollution in the environment, the determination of heat sources in heat conduction, etc. Consequently, inverse source problems for the heat equation have attracted considerable interest, see e.g. [1–8]. In all these studies the source function is sought as a function of either space or time. The reason for this restriction is the lack of uniqueness of a solution in the general case when the source depends on both space and time [9]. That is why inverse problems for finding sources depending on various/several variables are of great interest. For example, it is possible to restore uniqueness if we seek the source as a linear combination of point sources [10], or as an additive [11], or multiplicative [12], expression of separate time and space-dependent continuous components.

The objective of this paper is to determine heat source functions depending on both space and time, which are the sum of two unknown components depending separately on space and time, with known weights depending on time and space, respectively. The additional measurements/overspecified conditions are given by integral observations of the temperature over space and time. This is particularly advantageous in practical applications where local point or instant temperature measurements contain very large errors and then the use of non-local average measurements appears more realistic and reliable.

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The inverse problem is linear, but ill-posed. The local existence and uniqueness of a classical solution in Hölder spaces are established in Section 2, but more importantly this novel inverse formulation based on non-local average integral observations rather than local space or time point measurement enables the development of a weak solution theory for which variational methods are at hand, as developed in Section 3. The discretization of the direct and adjoint problems is based on the finite element method (FEM) which is briefly discussed in Section 4. The iterative conjugate gradient method (CGM) employed for minimizing the least-squares gap between the measured and computed data is also presented in Section 4. As expected, since the solution of the inverse problem does not depend continuously on the input data, regularization needs to be enforced in order to obtain a stable solution. This is performed by either stopping the CGM iteration at a threshold given by the discrepancy principle, or by penalizing the least-squares functional with extra regularization terms. Numerical results obtained for several benchmark test examples are presented and discussed in Section 5. Finally, Section 6 gives the conclusions of the paper.

2. Mathematical formulation

In this paper, we consider the particular practical application of the inverse analysis in the search of the heat source distribution in a multi-dimensional conductor. The determination of this heat source distribution across the space and time solution domain has a significant importance on finding the characteristics and performances of the thermal field. Further, it also assists in the designing of new heat conducting devices with an improved performance.

Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, and T a given positive number. Denote $Q := \Omega \times (0, T]$, and $S := \partial\Omega \times (0, T]$. In [11], the problem of determining the right hand coefficients $f_1(x)$ and $f_2(t)$ in the Dirichlet problem

$$u_t = \Delta u + g_0(x, t) + f_1(x)g_1(t) + f_2(t)g_2(x), \quad (x, t) \in Q, \tag{2.1}$$

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{2.2}$$

$$u|_S = u_S, \tag{2.3}$$

from the two additional conditions

$$u(x_0, t) = h(t), \quad t \in [0, T], \tag{2.4}$$

$$\int_0^T u(x, t)dt = g(x), \quad x \in \bar{\Omega}, \tag{2.5}$$

where x_0 is a fixed point in Ω, has been considered. Based on the trivial identity $(f_1(x)+cg_2(x))g_1(t)+(f_2(t)-cg_1(t))g_2(x) = f_1(x)g_1(t)+f_2(t)g_2(x)$, where c is an arbitrary constant, one can see that problem (2.1)–(2.5) does not have a unique solution. However, if one imposes the additional condition that $f_1(x_0)$ is known then, under some conditions on the smoothness of the data and their compatibility, it can be established, see [11], that if T is small, then there exists a unique solution to the inverse problem. The aim of our paper is to solve this inverse source problem by a variational method. Since the pointwise measurement (2.4) cannot be defined in the usual weak form framework, we replace it by the integral measurement

$$l_1 u := \int_\Omega \omega_1(x)u(x, t)dx = h(t), \quad t \in (0, T), \tag{2.6}$$

where ω_1 is a given function.

We also assume that we have prescribed the quantity

$$\int_\Omega \omega_1(x)f_1(x)dx = C_0. \tag{2.7}$$

Then we have the following local uniqueness solvability theorem.

Theorem 2.1. *Suppose that the following conditions are satisfied:*

(A1) *Eq. (2.2) holds for* $x \in \overline{\Omega}$, *Eq. (2.3) holds on* $\overline{S}$ *and Eq. (2.6) holds for* $t \in [0, T]$*;*
(A2) $u_0 \in H^{2+\gamma}(\overline{\Omega}), g \in H^{2+\gamma}(\overline{\Omega}), u_s \in H^{2+\gamma,1+\gamma/2}(\overline{\Omega}), g_0 \in H^{\gamma}(\overline{\Omega}), h \in H^{1+\gamma/2}[0, T], \omega_1 \in H^{2+\gamma}(\Omega), \partial\Omega \in H^{2+\gamma}$, *where* $\gamma \in (0, 1)$*;*
(A3) $u_0|_{\partial\Omega} = u_s(\cdot, 0)|_{\partial\Omega},\ g|_{\partial\Omega} = \int_0^T u_s(\cdot, t)|_{\partial\Omega}dt,\ h(0) = \int_\Omega \omega_1(x)u_0(x)dx,\ \int_\Omega \omega_1(x)g(x)dx = \int_0^T h(t)dt$*;*
(A4) $\int_0^T g_1(t)dt \neq 0,\ \int_\Omega \omega_1(x)g_2(x)dx \neq 0,\ g_1(t)/\int_0^T g_1(\tau)d\tau \geq 0,\ t \in [0, T]$.

Then for sufficiently small $T > 0$ *there exists a unique solution* $(f_1, f_2, u) \in H^{\gamma}(\overline{\Omega}) \times H^{\gamma/2}[0, T] \times H^{2+\gamma,1+\gamma/2}(\overline{Q})$ *of the inverse problem given by Eqs. (2.1)–(2.3) and (2.5)–(2.7).*

For the definition of the above Hölder spaces involved, see [13, p. 7].