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A partition of unity method for the displacement obstacle problem of clamped Kirchhoff plates ☆

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ABSTRACT

A partition of unity method for the displacement obstacle problem of clamped Kirchhoff plates is considered in this paper. We derive optimal error estimates and present numerical results that illustrate the performance of the method.

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1. Introduction

Let Ω be a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, $f \in L_2(\Omega)$, $g \in H^4(\Omega)$, and $\psi_1, \psi_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ be two obstacle functions such that

$$\psi_1 < \psi_2 \quad \text{in } \Omega \quad \text{and} \quad \psi_1 < g < \psi_2 \quad \text{on } \partial\Omega. \tag{1.1}$$

Consider the following problem: find $u \in H^2(\Omega)$ such that

$$u = \underset{v \in K}{\operatorname{argmin}}\, G(v), \tag{1.2}$$

where

$$K = \{v \in H^2(\Omega) : v - g \in H_0^2(\Omega),\ \psi_1 \leq v \leq \psi_2 \text{ on } \Omega\}, \tag{1.3}$$

$$G(v) = \frac{1}{2}a(v, v) - (f, v), \tag{1.4}$$

$$a(v, w) = \int_\Omega \nabla^2 v : \nabla^2 w\, dx, \qquad (f, v) = \int_\Omega f v\, dx \tag{1.5}$$

and $\nabla^2 v : \nabla^2 w = \sum_{i,j=1}^2 v_{x_i x_j} w_{x_i x_j}$ is the (Frobenius) inner product of the Hessian matrices of v and w.

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Since K is a nonempty closed convex subset of $H^2(\Omega)$ and $a(\cdot,\cdot)$ is symmetric and coercive on $H_0^2(\Omega)$ which contains the set $K-K=\{v-w : v, w\in K\}$, it follows from the standard theory [1–4] that (1.2) has a unique solution $u\in K$ characterized by the following variational inequality:

$$a(u, v-u)\geq (f, v-u)\quad \forall v\in K. \tag{1.6}$$

The convergence of finite element methods for second order obstacle problems was investigated in [5–7], shortly after it was shown in [8] that the solutions for such obstacle problems belong to $H^2(\Omega)$ under appropriate regularity assumptions on the data. This full elliptic regularity allows the complementarity form of the variational inequality (in the strong sense) to be used in the convergence analysis.

In contrast, the solutions of fourth order obstacle problems do not belong to $H^4_{loc}(\Omega)$ even if all the data are smooth [9]. It was shown in [10,11,9] that the solution u of (1.2)/(1.6) belongs to $H^3_{loc}(\Omega)\cap C^2(\Omega)$ under the assumptions above on f, g, ψ_1 and ψ_2. Since the obstacles are separated from each other and from the displacement boundary condition (cf. (1.1)), we have $\Delta^2 u = f$ near $\partial\Omega$. Therefore it follows from the elliptic regularity theory for the biharmonic operator on polygonal domains [12–15] that $u\in H^{2+\alpha}(\mathcal{N})$ for some $\alpha\in(\frac{1}{2},2]$ in an open neighborhood $\mathcal{N}$ of $\partial\Omega$. The elliptic regularity index α is determined by the interior angles of Ω and we can take α to be 1 for convex Ω. Thus the solution u of (1.2)/(1.6) belongs to $H^{2+\alpha}(\Omega)\cap H^3_{loc}(\Omega)\cap C^2(\Omega)$ in general.

This lack of $H^4_{loc}(\Omega)$ regularity means that the complementarity form of (1.6) only exists in a weak sense, and the convergence analysis based on the second order approach would only lead to suboptimal error estimates.

A new convergence analysis for finite element methods for (1.2)/(1.6) that does not rely on the complementarity form of the variational inequality (1.6) was proposed in [16], where optimal convergence was established for C^1 finite element methods, classical nonconforming finite element methods, and C^0 interior penalty methods for clamped plates ($g=0$) on convex domains. The results in [16] were subsequently extended to general polygonal domains and general Dirichlet boundary conditions for a quadratic C^0 interior penalty method [17] and a Morley finite element method [18]. The goal of this paper is to extend the results in [17,18] to a partition of unity method (PUM) for plates [19,20].

The rest of the paper is organized as follows. We introduce the partition of unity method in Section 2 and carry out the convergence analysis in Section 3. Numerical results are reported in Section 4, followed by some concluding remarks in Section 5.

2. A partition of unity method

We begin with the construction of the approximation space V_h in Section 2.1 and define an interpolation operator from $H^2(\Omega)$ into V_h in Section 2.2. The discrete obstacle problem is given in Section 2.3. We refer the readers to [21,22] for various aspects of generalized finite element methods.

2.1. Construction of the approximation space

The approximation space is based on partition of unity by flat-top functions [23–25].

2.1.1. Partition of unity

Let ϕ be the C^1 piecewise polynomial function given by

$$\phi(x)=\begin{cases}\phi^L(x):=(1+x)^2(1-2x) & \text{if } x\in[-1,0]\\ \phi^R(x):=(1-x)^2(1+2x) & \text{if } x\in[0,1]\\ 0 & \text{if } |x|\geq 1,\end{cases}$$

which enjoys the partition of unity property that

$$\phi^L(x-1)+\phi^R(x)=1\quad \text{for } 0\leq x\leq 1. \tag{2.1}$$

We define a flat-top function ψ_δ by

$$\psi_\delta(x)=\begin{cases}\phi^L\left(\dfrac{x-(-1+\delta)}{2\delta}\right) & \text{if } x\in[-1-\delta,-1+\delta]\\ 1 & \text{if } x\in[-1+\delta,1-\delta]\\ \phi^R\left(\dfrac{x-(1-\delta)}{2\delta}\right) & \text{if } x\in[1-\delta,1+\delta]\\ 0 & \text{if } x\notin[-1-\delta,1+\delta].\end{cases}$$

Here δ is a small number that controls the width of the flat-top part of this function where $\psi_\delta=1$.

For ease of presentation we take Ω to be a rectangle $(a,b)\times(c,d)$. But the construction and analysis can be extended to other domains (cf. Remark 2.4 and Examples 4.4 and 4.5 in Section 4).

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