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On the superconvergence patch recovery techniques for the linear finite element approximation on anisotropic meshes

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ABSTRACT

We provide in this paper an analysis on the superconvergence patch recovery (SPR) techniques for the linear finite element approximation based on adaptively refined anisotropic meshes in two dimensions. These techniques include the gradient recovery based on local weighted averaging, the recovery based on local L^2-projection, and the recovery based on least square fitting. The last one leads to the Zienkiewicz–Zhu type error estimators popular in engineering communities. Based on the superconvergence result for anisotropic meshes established recently in Cao (2013), we prove that all three types of SPR techniques produce super-linearly convergent gradients if the meshes are quasi-uniform under a given metric and each pair of adjacent elements in the meshes form an approximate parallelogram. As a consequence, the error estimators based on the recovered gradient are asymptotically exact. These results provide a theoretical justification for the extraordinary robustness and accuracy observed in numerous applications for the recovery type error estimators on anisotropic meshes.

1. Introduction

For problems with local anisotropic features, such as boundary and internal layers, finite element (FE) approximation based on adaptive anisotropic meshes can be much more efficient than the one based on shape regular meshes. Over the last decade, there has been a growing interest in the analysis and application on the finite element method (FEM) based on anisotropic meshes, see, e.g., [1–13]. An important component of the adaptive FE solution process is the a-posteriori error estimation, which is used to assess the accuracy of the numerical solutions and to guide the adaptive mesh refinements. There are mainly two types of a-posteriori error estimate techniques, residual based estimates and recovery based estimates. Both types of techniques are well established for the FE approximation on shape regular meshes, see, e.g., the monographs by Ainsworth and Oden [14], Babüska and Strouboulis [15], Wahlbin [16], and Zhu [17]. For the FE approximation on anisotropic meshes, there have been much research on the residual based error estimate techniques in recent years. For instance, Kunert and Nicaise [10,11] proved both upper and lower bounds for various residual based error estimators for two and three dimensional problems. Formaggia [18] studied the residual based error estimators for advection–diffusion–reaction problems and Stokes problems. Creusé and Nicaise [4], and Houston [9] provided an analysis on the residual based error estimators for the discontinuous Galerkin method on anisotropic meshes.

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On the other hand, there have also been many reports on the numerical study of the recovery type error estimates on anisotropic meshes, including the prominent Zienkiewicz–Zhu (ZZ) type [19,20] error estimators in particular [3,5,13, 21–25]. Numerous examples and applications demonstrate that recovery type error estimators behave extraordinary well even at the presence of highly anisotropic elements. They are not only reliable and efficient, but also often asymptotically exact. These features make recovery type estimators a favorable choice for FE practitioners in engineering communities. However, there has been very few theoretical work justifying these good behaviors in the case of anisotropic adaptive meshes. It is well-known that for the FEM based on shape regular meshes, superconvergence of the finite element solution plays an indispensable role in the analysis on the recovery type error estimators. This theory is still under development for anisotropic meshes. There have been a few superconvergence studies on the FE approximation on anisotropic meshes. But they mostly deal with uniform meshes or structured meshes [26–29]. Superconvergence for linear FE approximation on general adaptively refined anisotropic meshes has not been established until very recently in [30].

In [30] we considered the finite element approximation based on a class of meshes that are quasi-uniform under a given metric. By extending the concept of approximate parallelogram introduced in [31,32] for shape regular meshes to anisotropic meshes, we established the super-linear convergence of the finite element solution to the interpolation of the exact solution in H^1-norm. We also obtained the super-linear convergence for a gradient recovery based on the global L^2-projection of gradient of the finite element solution. It implies that the error estimator based on the global L^2-projection recovery is asymptotically exact for the adaptive anisotropic meshes.

The purpose of this paper is to provide an analysis on the superconvergence patch recovery (SPR) techniques, which are based on local instead of global operations, for the linear FEM on anisotropic meshes in two dimensions. More precisely, we study three types of gradient recovery in the SPR family. They include the recovery based on local weighted averaging, the recovery based on local L^2-projection, and the recovery based on least square fitting. The third one leads to the well-known ZZ-type error estimators. Based on the superconvergence result established in [30] and following similar ideas as in [33,34], we prove that all three types of SPR recovery techniques produce super-linearly convergent gradient if the meshes are quasi-uniform under a given metric and each pair of adjacent elements in the meshes form an approximate parallelogram. As a consequence, the error estimators based on the SPR gradient recovery are asymptotically exact. These results provide a theoretical justification for the extraordinary robustness and accuracy observed in numerous applications for the recovery type error estimators on anisotropic meshes.

A sketch of the paper is as follows: In Section 2 we first describe the FE approximation based on a class of anisotropic meshes that are quasi-uniform under a metric. Then we summarize the superconvergence of the FE solution established in [30]. In particular, we recall the concept of approximate parallelograms introduced there, which is also essential for our analysis of the SPR techniques. In Section 3 we develop the analysis on the recovery techniques based on local weighted averaging, local L^2-projection, and least square fitting, separately. We present in Section 4 a numerical example to demonstrate the superconvergence of the recovery techniques and the asymptotic exactness of the error estimators. And we conclude the paper with some comments in Section 5.

Throughout this paper, we use c to represent a general positive constant independent of the mesh and the functions involved, and use "$\simeq$" to represent the quantities on each side of it are equivalent with equivalence constants independent of the mesh and functions involved. For vectors we typically use $|\cdot|$ for their Euclidean norms, and for matrices we use $\|\cdot\|$ for their 2 -norms. For functions defined on domain D, we use $\|\cdot\|_D$ and $\|\cdot\|_{1,D}$ to represent, respectively, their L^2-norm and H^1-norm over D. When D is the entire domain for the PDE, we may omit the subscript D in the norms.

2. Model PDE and its FE approximation based on anisotropic meshes

We consider the following homogeneous Dirichlet problem of a second order elliptic equation:

$$\begin{cases} -\nabla\cdot(A\nabla u + \boldsymbol{b}u) + du = f, & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where A is a positive definite constant matrix, $\boldsymbol{b}$, d, and f are suitably smooth functions. Eq. (1) is assumed to be strongly elliptic.

FE approximation: Note that for discretization involving anisotropic meshes, the element diameter is no longer a proper parameter for describing the asymptotic behaviors, since each element can have much different length scales in different directions. Instead, the total number N of elements should be used to characterize the fineness of the discretization. Thus we use $\{\mathcal{T}_N\}$ to represent a family of triangulations of Ω satisfying the basic requirement that the intersection of the closures of any two elements is either the empty set, a point, or an entire edge. Define S_N be the space of continuous piecewise linear functions based on partition $\mathcal{T}_N$. $V_N = S_N \cap H_0^1(\Omega)$. The finite element method for solving (1) is to find the approximate solution $u_N \in V_N$ satisfying

$$\int_\Omega [(A\nabla u_N)\cdot\nabla v + u_N\,(\boldsymbol{b}\cdot\nabla v) + d\,u_N v] = 0, \quad \forall v \in V_N. \tag{2}$$

In order to better describe and control the anisotropic mesh features, such as element sizes, aspect ratios, and alignment directions, we restrict our analysis to a class of meshes that are quasi-uniform under a given metric. Let M be a continuous