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Goal-oriented a posteriori error estimation for finite volume methods

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ABSTRACT

A general framework for goal-oriented a posteriori error estimation for finite volume methods is developed. The framework does not rely on recasting finite volume methods as special cases of finite element methods, but instead directly determines error estimators from the discretized finite volume equations. Thus, the framework can be applied to arbitrary finite volume methods. It also provides the proper functional settings to address well-posedness issues for the primal and adjoint problems. Numerical results are presented to illustrate the validity and effectiveness of the a posteriori error estimates and their applicability to adaptive mesh refinement.

1. Introduction

Finite volume methods have become increasingly popular due to their intrinsic conservative properties and their capability in dealing with complex domains; see, e.g. [1–3]. Hence, a posteriori error estimates for finite volume methods are important as they aid in error control and improve the overall accuracy of numerical simulations. A posteriori error estimates also play a key role in the implementation of adaptive mesh refinement methods. In fact, the main motivation of the current work is to find simple and robust a posteriori error estimators to guide adaptive mesh refinements for finite volume methods in regional climate modeling [4–6].

The literature on goal-oriented a posteriori error analysis for the finite volume methods is slim compared to that for the finite element methods. A large part of the available works aim to derive a posteriori error bounds for approximate solutions in certain global energy norms; see, e.g. [7–14].

We are particularly interested in another type of a posteriori error estimates that are goal oriented, i.e., estimates of errors in certain quantities of interest. A goal-oriented error estimate is potentially very useful in assisting in error control. However, the literature on goal-oriented a posteriori error estimates for finite volume methods is even scarcer, probably due to the fact that finite volume methods do not naturally fit into variational frameworks. Insightful efforts have been made to address this challenge by exploiting the equivalence between certain finite volume methods and numerical schemes in variational forms such as the finite element methods or discontinuous Galerkin methods. For example, in [15],

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a goal-oriented a posteriori error estimate is presented for a special type of finite volume method that is equivalent to a Petrov–Galerkin variant of the discontinuous Galerkin method. In [16], an a posteriori error analysis is presented for cell-centered finite volume methods for the convection–diffusion problem by utilizing the equivalence between the finite volume methods and the lowest-order Raviart–Thomas mixed finite element method with a special quadrature. However, the applicability of this approach is limited for two reasons. First, on many grids on which finite volume schemes are constructed, there are no quadrature rules known, and thus implementing finite element or discontinuous Galerkin schemes on these grids is impractical. An example of such grids is hexagonal Voronoi grid [17]. The other reason is that finite volume methods for real-world problems are often sophisticated in themselves, and it is often not clear, to say the least, how to establish a connection to schemes in variational forms; but see also [18].

In this work, we aim to derive a general functional analytic framework for a posteriori error estimation for arbitrary finite volume methods. The idea is to derive a posteriori error estimators at the partial differential equation level in an appropriate functional setting. Since most differential operators are unbounded in the suitable functional spaces, we employ the relevant concept from functional analysis [19] to rigorously derive the adjoint operators. This approach does not require the differential equations or the numerical schemes to be recast in variational form nor do they rely on connecting a finite volume method with a finite element method. The approximate solutions produced by finite volume methods are simply taken as inputs to the a posteriori error estimator. Because the a posteriori error estimation is independent of the exact form of the finite volume method, it can be applied to arbitrary finite volume methods.

Pierce and Giles [20] performed a posteriori error estimation on the one-dimensional nonlinear Euler equation by analytically deriving the adjoint equations and their boundary conditions using the continuous Lagrangian multiplier (see also [21]). Giles et al. [22] dealt with the two-dimensional Euler equations using the direct integration-by-parts approach; see also [23]. The approach of [22] represents the essence of the framework that we advocate here, but [22] lacks both in details and mathematical rigor. Hence, by laying down an abstract framework for this approach, we hope to place it in an appropriate functional setting so that more fundamental questions, e.g. well-posedness of the primary and adjoint equations, can be addressed. We should note that Giles and Süli [24] similarly sought to lay down an abstract framework for a posteriori error estimation at the continuous level, but they did not handle the hyperbolic case correctly. In Section 8.2 of [24] for a hyperbolic case, it is stated that the solution of the primary problem and the solution of the adjoint problem are drawn from the same function space (see Eqs. (8.9) and (8.11) of [24]). But, for hyperbolic problems, the solution of the primary problem and the solution of the adjoint problem are subjected to different differential operators, and satisfy different boundary conditions, and generally speaking, they reside in different function spaces.

In Galerkin finite element or discontinuous Galerkin methods, the difference $u-u^{\#}$ between the exact solution u and the approximate solution $u^{\#}$ is orthogonal to the test function space V_h. For this reason, a posteriori error estimation for these methods requires that the adjoint solution ϕ be sought in a space $V_{h'}$ larger than V_h. This restriction does not apply in our approach due to the very fact that finite volume schemes do not naturally fit into variational forms, though for the sake of accuracy, it may be advantageous to seek the adjoint solution with a higher-order scheme. This point will be made clear in the next section.

It has been pointed out that the well-posedness issue for the adjoint equation remains challenging and open in many cases; see, e.g., [15]. A byproduct of our approach is that, because the adjoint problem is naturally posed in an appropriate functional setting, its well posedness can be dealt with by the abundant analytical tools of standard partial differential equation theories; again, see [15].

The rest of the paper is organized as follows. At the beginning of the next section, we present our approach for a posteriori error estimation for finite volume methods in a general functional analytical framework. It is followed by a numerical example demonstrating the error estimates. In Section 3, we present an application of the a posteriori error estimation to adaptive mesh refinement. We conclude with some remarks in Section 4.

2. A posteriori error estimates for finite volume methods

2.1. Abstract framework

Let H denote a Hilbert space endowed with the norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let L denote an unbounded linear operator in H with domain $D(L)$ dense in H. The primal problem we deal with is succinctly formulated in this functional setting as:

for each $f \in H$, find $\boldsymbol{u} \in D(L)$ such that

$$L\boldsymbol{u} = f. \tag{1}$$

We consider time-dependent problems so that the operator L usually takes the form

$$L = \frac{\partial}{\partial t} + A,$$

where A represents a linear differential spatial operator that is usually also unbounded in a respective function space. In this work, we assume that the primal problem (1) is well-posed, i.e., it possesses a unique solution.