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Multivariate spline approximation of the signed distance function

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Dedicated to Professor Benyu Guo on the occasion of his 70th Birthday

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ABSTRACT

The signed distance function can effectively support many geometry processing tasks such as decimates, smoothing and shape reconstruction since it provides efficient access to distance estimates. In this paper, we present an adaptive method to approximate the signed distance function of a smooth curve by using polynomial splines over type-2 triangulation. The trimmed offsets are also studied.

1. Introduction

The signed distance function (SDF) of a closed curve Γ in the plane assigns to any point x the shortest distance between x and Γ, with a positive sign if x is inside Γ and a negative one otherwise [1]. The SDF is widely used to deal with the problems in image processing, geometric computing, surface reconstruction, etc. [2,3].

Russo et al. [4] presented a method for the reconstruction of the SDF in the context of level set methods. It was a modification of the algorithm which made use of the PDE equation

$$\frac{\partial \phi}{\partial t} = \operatorname{sgn}(\phi^0)(1 - |\nabla \phi|),$$
$$\phi(x, 0) = \phi^0(x)$$

for the distance function introduced by Sussman et al. [5]. It was based mainly on the use of a truly upwind discretization near the interface. Belytschko et al. [6] developed an algorithm for smoothing the surfaces in finite element formulations of contact-impact. The smoothed signed distance functions were constructed by a moving least-squares approximation with a polynomial basis.

There is a close relationship between SDF and the level set method. For example, the distance function preservation has several advantages in the geometric and numerical point of view [7–9]. Song et al. [1,10] presented a method to approximate the SDF of a smooth curve or surface and compute the medial axis by using polynomial splines over hierarchical T-meshes (PHT splines); the theory about PHT splines refers to Deng et al. [11].

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Offset curve is defined as the locus of the points which are at constant distance along the normal from the given curve. It is used as the CNC machine tool path in the CAD/CAM. There are many literatures dealing with various aspects of offset curves [12,13]. It is well known that the offsetting may produce curves with self-intersections, which have to be detected and eliminated in applications. After eliminating these unwanted segments of the offset curve, trimmed offset curve is obtained [1].

In this paper, we propose a method to compute the approximation of the SDF based on the quasi-interpolation operator on the bivariate spline space $S_2^1(\Delta_{mn}^{(2)})$ and $S_2^1(\bar{\Delta}_{mn}^{(2)})$. This method is simple since the derivatives are not required. We can approximate the SDF using non-uniform spline space with the adaptive refinement. Moreover, this method can be applied to compute an implicit approximation of the trimmed offset curves of any given curve. The paper is organized as follows. In Section 2, we introduce the definition and properties of SDF. We recall some facts about the bivariate spline space $S_2^1(\Delta_{mn}^{(2)})$ and $S_2^1(\bar{\Delta}_{mn}^{(2)})$, and their quasi-interpolation operators in Section 3. In Section 4, we present a method to construct the approximation of the SDF by bivariate splines. Some examples are provided in Section 5. Finally, we conclude the paper in Section 6.

2. The signed distance function

Consider a simple closed curve Γ in the Euclidean plane $\mathbb{R}^2$. The curve Γ divides $\mathbb{R}^2$ into three parts: the interior Γ^+, the exterior Γ^-, and Γ. We define the distance function, and the signed distance function of Γ as follows:

Definition 1 (*[14]*)**.** Let Γ be a simple closed curve in the Euclidean plane $\mathbb{R}^2$. The distance function to Γ is the function $x \in \mathbb{R}^2 \mapsto d(x, \Gamma)$ defined by:

$$d(x, \Gamma) = \min_{y \in \Gamma} |x - y|. \tag{2.1}$$

Definition 2 (*[14]*)**.** Let Γ be a simple closed curve in the Euclidean plane $\mathbb{R}^2$. The signed distance function to Γ is the function $x \in \mathbb{R}^2 \mapsto u(x, \Gamma)$ defined by:

$$u(x, \Gamma) = \begin{cases} -d(x, \Gamma) & \text{if } x \in \Gamma^+, \\ 0 & \text{if } x \in \Gamma, \\ d(x, \Gamma) & \text{if } x \in \Gamma^-. \end{cases} \tag{2.2}$$

Distance functions have a kink at the interface where $d = 0$ is a local minimum, causing problems in approximation derivatives on or near the interface. On the other hand, SDFs are monotonic across the interface and can be differentiated there with significantly higher confidence.

SDFs share all the properties of implicit functions. In addition, there are a number of new properties that only SDFs possess. The main property of SDF is

$$|\nabla u| = 1. \tag{2.3}$$

Eq. (2.3) is true only in a general sense. It is not true for points that are equidistant from at least two points on the interface. This set of points is often called a skeleton in the literature and is a zero measure set.

Definition 3 (*[15]*)**.** The set $\{x \in \mathbb{R}^2 \mid$ there exist at least two distinct points y and z in Γ such that $\|x - y\| = \|x - z\| = d(x, \Gamma)\}$ is called the skeleton of Γ.

Some interesting properties of SDF are listed below (cf. [14]):

(1) The unit normal vector:

$$\vec{n} = \frac{\nabla u}{|\nabla u|} = \frac{\nabla u}{1} = \nabla u.$$

(2) Mean curvature:

$$\kappa = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = \nabla \cdot (\nabla u) = \Delta u.$$

(3) Closest point on interface: given $x \in \mathbb{R}^2$, then the closest point y on Γ is

$$y = x - u(x)\vec{n}.$$

(4) Differentiability on Γ: u is differentiable on Γ almost everywhere.

(5) Convex: if Γ^+ is a convex region, then u is a convex function.