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On the convergence rate of dynamic iteration for coupled problems with multiple subsystems

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ABSTRACT

In multiphysical modeling coupled problems naturally occur. Each subproblem is commonly represented by a system of partial differential-algebraic equations. Applying the method of lines, this results in coupled differential-algebraic equations (DAEs). Dynamic iteration with windowing is a standard technique for the transient simulation of such systems. In contrast to the dynamic iteration of systems of ordinary differential equations, convergence for DAEs cannot be generally guaranteed unless some contraction condition is fulfilled. In the case of convergence, it is a linear one.

In this paper, we quantify the convergence rate, i.e., the slope of the contraction, in terms of the window size. We investigate the convergence rate with respect to the coupling structure for DAE and ODE systems and also for two and more subsystems. We find higher rates (for certain coupling structures) than known before (that is, linear in the window size) and give sharp estimates for the rate. Furthermore it is revealed how the rate depends on the number of subsystems.

1. Introduction

Downscaling and advanced physical modeling require multi-scale models in many technical simulations. E.g. in electrical engineering, circuits are coupled to refined models (for networks, semiconductors [1]) or to magnetic field models (electric machines [2]). This results in multiphysical problems.

Simulator coupling is a standard technique for the transient simulation of coupled multiphysics problems. At synchronization times data between simulators is exchanged. Each simulator computes the solution for a dedicated subsystem only. Then iteration of this process ensures the consistency of the overall solution. In applications, this is referred to as cosimulation, whereas in mathematics it is usually called dynamic iteration or waveform relaxation. However, many simulators that apply a cosimulation technique do not iterate and advance in time immediately after a new approximation is computed.

Our focus is on time integration. Thus we assume that spatially discretized models to be given. Typically this yields coupled systems of differential algebraic equations (DAEs) to be solved in time domain. Dynamic iteration of coupled ordinary differential equations (ODEs) always converges (super-)linearly [3], i.e., for time interval of length H (or the window size, see below) it holds

$$\left\|x^{k+1} - x^*\right\| \leq C(H) \left\|x^k - x^*\right\|,$$

where the contraction rate $C(H)$ usually satisfies $C(H) \to 0$ as $H \to 0$. In contrast, the dynamic iteration of DAE systems will not always converge. But convergence can be guaranteed if contraction conditions are fulfilled, see e.g. [4–6].

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The standard approach to derive the convergence result reads: first an error recursion is set up, which covers the solution process of all subsystem once; second, in order to apply Banach's fixed-point theorem, a contraction condition is derived from the recursion. Then convergence and stability follows.

In case of convergence, one has linear convergence with rate $\mathcal{O}(\sqrt{H})$ (error reduction per iteration) at most. Convergence is not only influenced by the coupling structure, but also by the order of computation [6,1] and by the actual dynamic iteration scheme employed.

A higher convergence rate can be obtained from certain coupling structures [7,1]. Also in applications (field–circuit coupling [2]), a higher rate was numerically observed.

The aim of this work is to derive an analytical background for the higher convergence rates and to present coupling structures, which guarantee convergence rates up to $\mathcal{O}(H^2)$. To this end, we apply the strategy for the error recursions from previous works [6,7,1] to systems with a refined structural analysis. Although Jacobi-type of dynamic iteration is quite popular (high potential of parallelization), we will investigate Gauss–Seidel-type iteration schemes that lead to faster convergence for particular DAE problems, which are free of contraction conditions.

The paper is organized as follows. Section 2 introduces the notation of coupled systems and dynamic iteration. Section 3 summarizes the known procedure to derive the recursion estimates, contraction and convergence. Our contribution of refined structural analysis for coupled DAEs follows in Section 4. We investigate two and multiple coupled systems and derive the convergence rate for different couplings. In Section 5 an academic test problem is presented that verifies the theoretical results by experiments and the application to real-world problems from electrical engineering is discussed. Conclusions form the final section.

2. Coupled DAE systems and dynamic iteration schemes

We consider coupled initial-value problems (IVPs), which can be written in semi-explicit form (for each of the r subsystems):

$$\dot{\mathbf{y}}_i = \mathbf{f}_i(\mathbf{y}, \mathbf{z}), \qquad \mathbf{y} := (\mathbf{y}_1, \ldots, \mathbf{y}_r)^\top, \qquad \mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^{n_y} \tag{1a}$$

$$0 = \mathbf{g}_i(\mathbf{y}, \mathbf{z}), \qquad \mathbf{z} := (\mathbf{z}_1, \ldots, \mathbf{z}_r)^\top, \qquad \mathbf{z}(0) = \mathbf{z}_0 \in \mathbb{R}^{n_z} \tag{1b}$$

with $i = 1, \ldots, r$. Without loss of generality, this system is in autonomous form. This system is a split structure for an overall semi-explicit DAE system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{z})$, $0 = \mathbf{g}(\mathbf{y}, \mathbf{z})$ with $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r)^\top$ and $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_r)^\top$.

Assumption 1. Given the coupled DAE-problem (1).

(a) Let $\mathbf{f}$ and $\mathbf{g}$ be sufficiently often differentiable.
(b) Let consistent initial-values be given (i.e., $0 = \mathbf{g}(\mathbf{y}_0, \mathbf{z}_0)$).
(c) Let $\partial\mathbf{g}/\partial\mathbf{z}$ and $\partial\mathbf{g}_i/\partial\mathbf{z}_i$ be regular for each i with inverse ϕ_i.

In other words, by Assumption 1 the i-th subsystem is index-1 with respect to the variables $\mathbf{y}_i$ and $\mathbf{z}_i$ given all other variables ($i = 1, \ldots, r$) and the overall system is index-1. Hence $\mathbf{y}_i$ and $\mathbf{z}_i$ are internal variables of the i-th system and the remainder are its coupling variables. Furthermore there is a unique solution $\mathbf{x} = (\mathbf{y}, \mathbf{z})^\top \in C^1([0, t_e], \mathbb{R}^{n_y}) \times C([0, t_e], \mathbb{R}^{n_z})$ with $\mathbf{y} : [0, t_e] \to \mathbb{R}^{n_y}$, $\mathbf{z} : [0, t_e] \to \mathbb{R}^{n_z}$ [8]. With the trivial constraint $\mathbf{g}_i \equiv 0$ and dimension $n_{z_i} = 0$ (i.e., $\mathbf{z}_i(t) \in \mathbb{R}^{n_{z_i}}$), the split system (1) may include also ODE subsystems.

We aim at computing a sufficiently accurate approximation $\tilde{\mathbf{x}} := (\tilde{\mathbf{y}}, \tilde{\mathbf{z}})^\top : [0, t_e] \to \mathbb{R}^{n_y} \times \mathbb{R}^{n_z}$ of the unique $\mathbf{x}$ for (1). This approximation is represented by a continuous waveform in our analysis. In practice it is represented by sampled points stemming from a sufficiently accurate numerical time stepping procedure.

Dynamic iteration schemes compute approximations to the subsystems separately, while an outer iteration loop can guarantee convergence towards the desired unique solution. Thus each subsystem may invoke a dedicated solver to respect the corresponding structures like stiffness, activity, definiteness etc.

For the analysis, we formalize the iteration procedure. It is commonly performed on so called (time-)windows $[t_n, t_{n+1}]$ with $0 = t_0 < t_1 < t_2 < \cdots < t_N = t_e$ with window size $H_n := t_{n+1} - t_n$. Given a numerical approximation $\tilde{\mathbf{x}}$ on $[0, t_n]$, a dynamic iteration defines the approximations on the next window:

$$(\tilde{\mathbf{y}}, \tilde{\mathbf{z}})|_{(t_n, t_{n+1}]} \in C_n^{1,0} \quad \text{with } C_n^{1,0} := C^1((t_n, t_{n+1}], \mathbb{R}^{n_y}) \times C((t_n, t_{n+1}], \mathbb{R}^{n_z})$$

via an extrapolation $\Phi_n : C_{n-1}^{1,0} \to C_n^{1,0}$ and a number of iterations $\Psi_n : C_n^{1,0} \to C_n^{1,0}$: (see e.g. [6])

$$\Phi_n : \begin{pmatrix} \tilde{\mathbf{y}}|_{[t_{n-1}, t_n]} \\ \tilde{\mathbf{z}}|_{[t_{n-1}, t_n]} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\mathbf{y}}_n^{(0)} \\ \tilde{\mathbf{z}}_n^{(0)} \end{pmatrix} \qquad \Psi_n : \begin{pmatrix} \tilde{\mathbf{y}}_n^{(k-1)} \\ \tilde{\mathbf{z}}_n^{(k-1)} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\mathbf{y}}_n^{(k)} \\ \tilde{\mathbf{z}}_n^{(k)} \end{pmatrix} := \Psi_n \begin{pmatrix} \tilde{\mathbf{y}}_n^{(k-1)} \\ \tilde{\mathbf{z}}_n^{(k-1)} \end{pmatrix}.$$

Subscript n refers to the time window and superscript $k = 1, \ldots, k_n$ denotes the iteration count of the dynamic iteration Ψ_n (with k_n finite). To solve the split DAE (1) for $\tilde{\mathbf{y}}_n = (\tilde{\mathbf{y}}_{1,n}, \ldots, \tilde{\mathbf{y}}_{r,n})^\top$, $\tilde{\mathbf{z}}_n = (\tilde{\mathbf{z}}_{1,n}, \ldots, \tilde{\mathbf{z}}_{r,n})^\top$, operator Ψ_n maps given approximations (index $(k-1)$) to new ones $\tilde{\mathbf{y}}^{(k)}$, $\tilde{\mathbf{z}}^{(k)}$ of the decoupled initial-values problems. To this end, the right-hand