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Characteristic boundary conditions in the lattice Boltzmann method for fluid and gas dynamics

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ABSTRACT

For numerically solving fluid dynamics problems efficiently one is often facing the problem of having to confine the computational domain to a small domain of interest introducing so-called non-reflecting boundary conditions (NRBCs).

In this work we address the problem of supplying NRBCs in fluid simulations in two space dimensions using the lattice Boltzmann method (LBM): so-called characteristic boundary conditions are revisited and transferred to the framework of lattice Boltzmann simulations.

Numerical tests show clearly that the unwanted unphysical reflections can be reduced significantly by applying our newly developed methods. Hereby the key idea is to transfer and generalize Thompson's boundary conditions originally developed for the nonlinear Euler equations of gas dynamics to the setting of lattice Boltzmann methods. Finally, we give strong numerical evidence that the proposed methods possess a long-time stability property.

1. Introduction

For fluid simulations, the *lattice Boltzmann method* (LBM) has been proven to be a quite flexible tool [1–3]. Its ease of implementation and its applicability to complex flows (including multicomponent flow, multiphase flow, obstacles, complex physical interaction such as fluid structure interaction) make this method extremely attractive for real-world simulations.

To enable an efficient numerical simulation, often a small bounded simulation domain is needed. Such a situation is obtained by confining the domain correspondingly. Thereby the physical boundaries are supplemented with some additional, artificial boundaries, where numerical conditions have to be assigned to the state variables. Ideally, these boundaries and these boundary conditions shall have no influence on the simulation result, i.e., the interaction of the artificial boundary with numerical quantities shall at least be below the discretization error of the interior scheme. This is exactly the aim of absorbing or *non-reflecting boundary conditions* (NRBCs).

Several studies have been made on NRBCs for direct solvers. Starting from the pioneering work for absorbing boundary conditions for wave equations by Engquist and Majda [4], *characteristic boundary conditions* (CBCs) in the field of nonlinear hyperbolic equations were developed by Hedstrom [5] and Thompson [6], which are non-reflecting. Kröner [7] derived approximate, exact absorbing boundary conditions for the linear Euler equations. Poinsot and Lele [8] derived NRBCs for the Navier–Stokes equations.

In a different fully discrete approach, Wilson [9] and later Rowley and Colonius [10] derived for the linear Euler equations the NRBCs directly for the chosen numerical scheme. This approach has the advantage that these discrete boundary

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conditions are already perfectly adapted to the interior scheme resulting in higher accuracy and better stability properties compared to the previous approaches. For a concise review article on absorbing boundary conditions for hyperbolic systems we refer the interested reader to [11]. We remark that a closely related question is the construction of so-called *far field boundary conditions* that are optimized to numerically approximate the stationary solution of the hyperbolic systems, cf. [12].

Let us emphasize that the situation of NRBCs for the LBM is completely different. Only a few studies have been made on this subject, cf. [13] for a comparison of different approaches in an aeroacoustic application. Recently, Najafi-Yazdi and Mongeau [14] developed an absorbing layer boundary condition, based on the *perfectly matched layer* (PML) concept. Similarly, Tekitek et al. [15], proposed a lattice Boltzmann scheme modeling the PML of Bérenger. Another approach was used by Izquierdo and Fueyo [16], who solved a system of differential equations and obtained a Dirichlet condition with non-reflecting properties. Their procedure is an approximate method to the *Navier–Stokes characteristic boundary condition* (NS-CBC) by Poinsot and Lele [8]. This system contains only one dimensional information at the artificial boundaries, so their condition can be seen as an implementation of simplified Thompson's boundary conditions [6] where higher spatial derivatives are neglected. The aim of this work is to extend these known CBCs for the LBM application in two respects: (a) inclusion of more spatial information (derivatives) at the artificial boundary and (b) enable possibly smaller reflection rates.

To this end, this article is structured as follows. In Section 2 we present a short introduction to the LBM and also explain briefly the construction of a boundary condition within this framework. Section 3 is devoted to a description of NRBCs. Here the conditions are described continuously and are therefore formulated independently of the used numerical method. Then, in Section 4 we explain how the boundary conditions of Section 3 are implemented within the fully discrete lattice Boltzmann context in two dimensions. In the last section we present our numerical results of three test cases for the NRBCs and finally we conclude.

2. The lattice Boltzmann method

The Boltzmann equation describes the evolution of the single particle distribution function $f(\vec{x}, \vec{\xi}, t)$:

$$\frac{\partial f(\vec{x}, \vec{\xi}, t)}{\partial t} + \vec{\xi} \cdot \nabla f(\vec{x}, \vec{\xi}, t) = Q(f),$$

in terms of the space coordinate $\vec{x} \in \mathbb{R}^d$, molecular velocity $\vec{\xi} \in \mathbb{R}^d$, time $t > 0$ and collision term $Q(f)$. From this mesoscopic description macroscopic quantities like the mass density ρ and the fluid velocity $\vec{u}$ are obtained by computing moments of f.

The LBM can be regarded as a special discretization of the Boltzmann equation [17], where the molecular velocity space $\vec{\xi} \in \mathbb{R}^d$ is restricted to a finite set of given velocities $\vec{c}_i \in \mathbb{R}^d,\ i = 0, \ldots, n_v$. Next, for a given time step size Δt the spatial discretization is obtained with the velocity set by $\Delta\vec{x} = \vec{c}_i \Delta t$. That is, in the time period Δt particles move from one lattice site $\vec{x}$ to a neighboring site $\vec{x} + \Delta\vec{x}$. Commonly, in LBM the Boltzmann collision integral $Q(f)$ is approximated by a relaxation towards a local equilibrium $f^{(\mathrm{eq})}$. Here we use the popular BGK model [18], which is a single relaxation time model. This approach yields the following lattice Boltzmann equation [1]:

$$f_i(\vec{x} + \vec{c}_i \Delta t, t + \Delta t) - f_i(\vec{x}, t) = -\frac{\Delta t}{\tau}\left[f_i(\vec{x}, t) - f_i^{(\mathrm{eq})}(\vec{x}, t)\right], \tag{1}$$

which is an evolution equation for the discrete particle distribution f_i (corresponding to $\vec{c}_i$). The right-hand side, with the free relaxation parameter τ, is the discrete BGK-model for $Q(f)$. The quantities $f_i(\vec{x}, t)$ are called *populations*. In this work, we consider two space dimensions with 9 ($n_v = 8$) lattice velocities (D2Q9 model):

$$\vec{c}_0 = \vec{0}, \qquad \vec{c}_i = c \begin{pmatrix} \cos\left(\frac{\pi}{2}(i-1)\right) \\ \sin\left(\frac{\pi}{2}(i-1)\right) \end{pmatrix}, \quad i = 1, 2, 3, 4,$$

$$\vec{c}_j = c \begin{pmatrix} \sqrt{2}\cos\left(\frac{\pi}{2}\left(j - \frac{1}{2}\right)\right) \\ \sqrt{2}\sin\left(\frac{\pi}{2}\left(j - \frac{1}{2}\right)\right) \end{pmatrix}, \quad j = 5, 6, 7, 8.$$

Moreover, the local equilibrium $f_i^{(\mathrm{eq})}$ is given here by:

$$f_i^{(\mathrm{eq})}(\vec{x}, t) = w_i \rho(\vec{x}, t)\left[1 + \frac{3}{c^2}(\vec{c}_i \cdot \vec{u}(\vec{x}, t)) + \frac{9}{2c^4}(\vec{c}_i \cdot \vec{u}(\vec{x}, t))^2 - \frac{3}{2c^2}|\vec{u}(\vec{x}, t)|^2\right] \tag{2}$$

with the weights $w_0 = 4/9$, $w_{1-4} = 1/9$ and $w_{5-8} = 1/36$. The macroscopic quantities mass density and fluid velocity are computed in each lattice point by

$$\rho(\vec{x}, t) = \sum_{i=0}^{8} f_i(\vec{x}, t), \qquad \vec{u}(\vec{x}, t) = \frac{1}{\rho(\vec{x}, t)} \sum_{i=1}^{8} \vec{c}_i f_i(\vec{x}, t). \tag{3}$$