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An analysis of the Prothero–Robinson example for constructing new DIRK and ROW methods

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A B S T R A C T

This note analyses the order reduction phenomenon of diagonally implicit Runge–Kutta methods (DIRK methods) and Rosenbrock–Wanner methods (ROW methods) applied on the Prothero–Robinson example. New order conditions to reduce order reduction are derived, and a new third-order DIRK and ROW method is created. The new schemes are applied to the Prothero–Robinson example and on the semi-discretised incompressible Navier–Stokes equations. Numerical examples show that the new methods have better convergence properties than comparable methods.

1. Introduction

One possibility for solving stiff ordinary differential equations (ODEs), such as the example of Prothero and Robinson [1] or differential–algebraic equations, is to use Runge–Kutta methods [2,3]. Explicit methods may not be a good choice, because to generate a stable numerical solution a stepsize restriction should be accepted, i.e., the problem should be solved with very small time steps. Therefore it might be better to use implicit methods, such as for example Runge–Kutta methods, or linear-implicit methods such as Rosenbrock–Wanner methods. But in these cases convergence may not be achieved [2,3], i.e., the so-called order reduction phenomenon can be observed. The aim of this paper is to find further order conditions such that the example of Prothero and Robinson can be solved numerically with almost full convergence order. In [2], convergence results for implicit Runge–Kutta methods can be found where the so-called stage order plays an important role. Ostermann and Roche prove in [4] that implicit Runge–Kutta methods may have a fractional order of convergence for general linear ODEs. Similar results are presented for Rosenbrock–Wanner methods in [5]. As for diagonally implicit Runge–Kutta methods with non-zero diagonal entries, Rosenbrock–Wanner methods can have only stage order 1. That is the reason why Ostermann and Roche in [5] derive further order conditions for Rosenbrock–Wanner methods. Common methods such as ROS3P (see [6]) or ROS34PW2 (see [7]) satisfy these order conditions and have less order reduction if they are applied on stiff ODEs or the semi-discretised Navier–Stokes equations [7–10]. In a paper of Scholz [11], a different approach can be found for reducing the order reduction. A Rosenbrock–Wanner method satisfying the order conditions derived by Scholz [11] is the method RODASP [12].

Fully implicit Runge–Kutta methods may be ineffective for solving high-dimensional ODEs since they need a high computational effort, which can be reduced if diagonally implicit Runge–Kutta methods are used. Since the stage order is limited to 1 if all diagonal entries are non-zero, Cameron in [13] introduced the so-called quasi stage order. This concept is improved in [14], where the method SDIRK2 is derived. Often DIRK methods with an explicit first stage are used, since common methods

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have stage order 2 [15,16]. To reduce the order reduction, often order conditions for index-2 DAEs [17,18] are satisfied, as is shown in [19,20].

A question which this note tries to answer is the following: Can we construct a DIRK method with stage order 1 which converges with order ≥ 2 in the stiff case if applied, for example, to the ODE of Prothero and Robinson [1]?

Further considerations are motivated by the following observation. In [21], an embedded method for the fractional step-θ scheme (a special diagonally implicit Runge–Kutta method) is introduced, and it is shown that this method has convergence order 1. Solving the stiff example of Prothero and Robinson, it can be observed that the method converges with order 2 for certain time step sizes, although the known theoretical results suggest a convergence order of 1.

In this note, we consider diagonally implicit and Rosenbrock–Wanner methods and apply them to the Prothero–Robinson example. In Section 3, we consider the local error of these classes of methods in the non-stiff case and in the stiff case. We will see that we get further order conditions which are needed to decrease the order reduction. A third-order diagonally implicit Runge–Kutta method and a third-order Rosenbrock–Wanner method are created in Section 4, and finally we present some numerical results and apply our new methods to the Prothero–Robinson example and the incompressible Navier–Stokes equations.

2. Time discretisation

2.1. Rosenbrock–Wanner methods

First we consider an ODE of the form

$$\dot{\mathbf{u}} = \mathbf{F}(t, \mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{1}$$

A Rosenbrock–Wanner-method (ROW method) with s internal stages is given by

$$\mathbf{k}_i = \mathbf{F}\left(t_m + \alpha_i \tau_m, \tilde{\mathbf{U}}_i\right) + \tau_m J \sum_{j=1}^{i} \gamma_{ij}\mathbf{k}_j + \tau_m \gamma_i \dot{\mathbf{F}}(t_m, \mathbf{u}_m), \tag{2}$$

$$\tilde{\mathbf{U}}_i = \mathbf{u}_m + \tau_m \sum_{j=1}^{i-1} a_{ij}\mathbf{k}_j, \quad i = 1, \ldots, s,$$

$$\mathbf{u}_{m+1} = \mathbf{u}_m + \tau_m \sum_{i=1}^{s} b_i \mathbf{k}_i, \tag{3}$$

where $J := \partial_{\mathbf{u}}\mathbf{F}(t_m, \mathbf{u}_m)$, α_{ij}, γ_{ij}, b_i are the parameters of the method, and

$$\alpha_i := \sum_{j=1}^{i-1} \alpha_{ij}, \qquad \gamma_i := \sum_{j=1}^{i-1} \gamma_{ij}, \quad \gamma := \gamma_{ii} > 0,\ i = 1, \ldots, s.$$

If the parameters α_{ij}, γ_{ij}, and b_i are chosen appropriately, a sufficient consistency order can be obtained. Additional consistency conditions arise if J is only an approximation to $\partial_{\mathbf{u}}\mathbf{F}(t_m, \mathbf{u}_m)$, or if J is an arbitrary matrix. This class of methods is called W methods [3]. If a ROW method is applied to a semi-discretised partial differential equation, further order conditions should be satisfied to avoid order reduction; see [22].

The ROW method (2)–(3) requires the successive solution of s linear systems of equations with the same matrix $I - \gamma\tau_m J$. The right-hand side of the ith linear system of equations depends on the solutions of the first to the $(i-1)$th system. Thus, a main difference between ROW methods and implicit methods is that it is not necessary to solve a nonlinear system of equations in each discrete time but only a fixed number of linear systems of equations.

ROW methods have the advantage that they allow an easy implementation of an adaptive time step length control. Consider a ROW method of order $p \geq 2$. An adaptive time step control employs a second ROW method which has the coefficients a_{ij}, $\hat{b}_i$, and c_i, $i, j = 1, \ldots, s$, and order $p - 1$. The solution of the second method at t_{m+1} is given by

$$\hat{\mathbf{u}}_{m+1} = \mathbf{u}_m + \sum_{i=1}^{s} \hat{b}_i \mathbf{k}_i.$$

Now, the next time step τ_{m+1} is proposed to be

$$\tau_{m+1} = \rho \frac{\tau_m^2}{\tau_{m-1}} \left(\frac{TOL \cdot r_m}{r_{m+1}^2} \right)^{1/p}, \tag{4}$$

where $\rho \in (0, 1]$ is a safety factor, $TOL > 0$ is a given tolerance, and $r_{m+1} := \left\| \mathbf{u}_{m+1} - \hat{\mathbf{u}}_{m+1} \right\|$. This step size selection rule is called a PI controller, and it goes back to Gustafsson et al. [23]. For details on the numerical error and the implementation of automatic step length control, we refer to [2,24].

Next we apply the ROW method (2)–(3) to the Prothero–Robinson problem, i.e., on

$$\dot{u} = \lambda(u - \varphi(t)) + \dot{\varphi}(t), \quad u(0) = \varphi(0),\ \lambda < 0. \tag{5}$$