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A comparison of Filippov sliding vector fields in codimension 2

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ABSTRACT

We consider several possibilities on how to select a Filippov sliding vector field on a codimension 2 singularity surface Σ, intersection of two codimension 1 surfaces. We discuss and compare several, old and new, approaches, under the assumption that Σ is nodally attractive. Of specific interest is the selection of a smoothly varying Filippov sliding vector field. As a result of our analysis and experiments, the best candidates of the many possibilities explored are those based on the so-called barycentric coordinates. In the present context, one of these possibilities appear to be new.

1. Introduction

Our purpose in this paper is to discuss, and compare, several possibilities on how to select a Filippov sliding vector field on a codimension 2 singularity surface Σ, which is itself the intersection of two codimension 1 singularity surfaces. We give a unifying framework within which to compare the various possibilities considered, and we will highlight and clarify important connections to methods that have proven useful in computer graphics and finite elements techniques.

In this section, we review the basic problem and set up notation. Then, in Sections 2 and 3 we look at different possibilities for Filippov sliding vector fields. For convenience, we classify different choices as being either *analytic–algebraic methods* or *geometric methods*; the distinction is only for convenience of introducing the methods, but the geometric methods we consider can in fact be interpreted as special choices of analytic methods. Finally, in Section 4 we see how one may generally reformulate the problem with respect to *sub-sliding* vector fields. In Section 5 we give our conclusions.

1.1. The problem and Filippov solutions

We are interested in piecewise smooth differential systems of the following type:

$$\dot{x} = f(x), \quad f(x) = f_i(x), \; x \in R_i, \; i = 1, \dots, 4, \tag{1.1}$$

with initial condition $x(0) = x_0 \in R_i$, for some i. Here, the $R_i \subseteq \mathbb{R}^n$ are open, disjoint and connected sets, and we may as well think that $\mathbb{R}^n = \overline{\bigcup_i R_i}$. Each f_i is smooth on R_i, $i = 1, \dots, 4$, and we will assume that each f_i is actually smooth in an open neighborhood of the closure of each R_i, $i = 1, \dots, 4$. (Strictly speaking, this last assumption may actually be not needed, but it simplifies some of the later exposition.)

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Table 1
Nodal attractivity.

Component	$i=1$	$i=2$	$i=3$	$i=4$
w_i^1	>0	>0	<0	<0
w_i^2	>0	<0	>0	<0

Clearly, from (1.1), the vector field is not properly defined on the boundaries of the R_i's. We are particularly interested in analyzing what happens in this case, under the scenario that solution trajectories are attracted towards these boundaries.

1.2. Codimension 1: attractivity, existence and uniqueness

The classical Filippov theory (see [1]) is concerned with the case of two regions separated by a surface Σ defined as the 0-set of a smooth scalar valued function h:

$$\begin{aligned} &\dot{x} = f_1(x), \quad x \in R_1, \quad \text{and} \quad \dot{x} = f_2(x), \quad x \in R_2, \\ &\Sigma := \{x \in \mathbb{R}^n : h(x) = 0\}, \quad h : \mathbb{R}^n \to \mathbb{R}, \end{aligned} \tag{1.2}$$

where ∇h is bounded away from 0 for all $x \in \Sigma$, hence near Σ. As in [1], we assume that h is a $\mathcal{C}^k$ function, with $k \geq 2$. Finally, without loss of generality, we label R_1 such that $h(x) < 0$ for $x \in R_1$, and R_2 such that $h(x) > 0$ for $x \in R_2$.

The interesting case is when trajectories reach Σ from R_1 (or R_2), and one has to decide what happens next. To answer this question, it is useful to look at the components of the two vector fields $f_{1,2}$ orthogonal to Σ:

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} \nabla h(x)^\top f_1(x) \\ \nabla h(x)^\top f_2(x) \end{bmatrix}, \quad x \in \Sigma. \tag{1.3}$$

Filippov theory is a first order theory (that is, based on nonvanishing w_i, $i = 1, 2$) providing an answer to this situation. We call Σ *attractive in finite time* if for some positive constant c, we have

$$\nabla h(x)^\top f_1(x) \geq c > 0 \quad \text{and} \quad \nabla h(x)^\top f_2(x) \leq -c < 0,$$

for $x \in \Sigma$. In this case, trajectories starting near Σ must reach it and remain there: *sliding motion*. Filippov proposal is to take as sliding vector field on Σ a convex combination of f_1 and f_2, $f_F := (1-\alpha)f_1 + \alpha f_2$, with α chosen so that $f_F \in T_\Sigma$ (f_F is tangent to Σ at each $x \in \Sigma$):

$$x' = (1-\alpha)f_1 + \alpha f_2, \quad \alpha = \frac{\nabla h(x)^\top f_1(x)}{\nabla h(x)^\top f_1(x) - \nabla h(x)^\top f_2(x)}. \tag{1.4}$$

1.3. Codimension 2: nodal attractivity

As we said, we are concerned with (1.1), where now the R_i's are (locally) separated by two intersecting smooth surfaces of co-dimension 1, $\Sigma_1 = \{x : h_1(x) = 0\}$ and $\Sigma_2 = \{x : h_2(x) = 0\}$, and we have $\Sigma = \Sigma_1 \cap \Sigma_2$. As before, we will assume that $\nabla h_1(x) \neq 0, x \in \Sigma_1$, $\nabla h_2(x) \neq 0, x \in \Sigma_2$, that $h_{1,2}$ are $\mathcal{C}^k$ functions, with $k \geq 2$, and further that $\nabla h_1(x)$ and $\nabla h_2(x)$ are linearly independent for x on (and in a neighborhood of) Σ.

We have four different regions R_1, R_2, R_3 and R_4 with the four different vector fields f_i, $i = 1, \ldots, 4$, in these regions:

$$\dot{x} = f_i(x), \quad x \in R_i, \; i = 1, \ldots, 4. \tag{1.5}$$

Without loss of generality, we can label the regions as follows:

$$\begin{aligned} &R_1 : f_1 \quad \text{when } h_1 < 0,\; h_2 < 0, \qquad R_2 : f_2 \quad \text{when } h_1 < 0,\; h_2 > 0, \\ &R_3 : f_3 \quad \text{when } h_1 > 0,\; h_2 < 0, \qquad R_4 : f_4 \quad \text{when } h_1 > 0,\; h_2 > 0. \end{aligned} \tag{1.6}$$

We further let (cf. with (1.3))

$$\begin{aligned} &w_1^1 = \nabla h_1^\top f_1, \qquad w_2^1 = \nabla h_1^\top f_2, \qquad w_3^1 = \nabla h_1^\top f_3, \qquad w_4^1 = \nabla h_1^\top f_4, \\ &w_1^2 = \nabla h_2^\top f_1, \qquad w_2^2 = \nabla h_2^\top f_2, \qquad w_3^2 = \nabla h_2^\top f_3, \qquad w_4^2 = \nabla h_2^\top f_4, \end{aligned} \tag{1.7}$$

and restrict to the case of Σ being *nodally attractive*, a condition characterized by the constraints on the sign of w^1 and w^2 expressed in Table 1, which are assumed to be valid on Σ and near it (uniformly away from 0).

According to the present setup, when x is near Σ, a trajectory through x will be attracted to Σ, and – upon reaching it – will be forced to remain on it (*sliding motion*).

Remark 1.1. Nodal attractivity of Σ is just one of several different sufficient conditions under which Σ will attract nearby trajectories. Arguably, nodal attractivity is the simplest of all these sufficient conditions and it serves as a fundamental benchmark to evaluate different means for obtaining a sliding vector field on Σ. A more comprehensive classification of attractivity conditions for Σ is in [2], and we are currently investigating the behavior of some of the methods examined