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Structure preserving integrators for solving (non-)linear quadratic optimal control problems with applications to describe the flight of a quadrotor

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ABSTRACT

We present structure preserving integrators for solving linear quadratic optimal control problems. The goal is to build methods which can also be used for the integration of nonlinear problems if they are previously linearized. The equations are solved using an iterative method on a fixed mesh with the constraint that at each iteration one can only use results obtained in previous iterations on that fixed mesh. On the other hand, this problem requires the numerical integration of matrix Riccati differential equations whose exact solution is a symmetric positive definite time-dependent matrix which controls the stability of the equation for the state. This property is not preserved, in general, by the numerical methods. We analyze how to build methods for the linear problem taking into account the previous constraints, and we propose second order exponential methods based on the Magnus series expansion which unconditionally preserve positivity for this problem and analyze higher order Magnus integrators. The performance of the algorithms is illustrated with the stabilization of a quadrotor which is an unmanned aerial vehicle.

1. Introduction

Nonlinear control problems have attracted the interest of researchers in different fields, e.g., the control of airplanes, helicopters, satellites, etc. [1–3] during the last years. While the extensively studied linear quadratic optimal control (LQ) problems can be used for solving simplified models, most realistic problems are inherently nonlinear. Furthermore, nonlinear control theory can improve the performance of the controller and enable tracking of aggressive trajectories [4].

Solving nonlinear optimal control problems requires the numerical integration of systems of coupled non-autonomous and nonlinear equations with boundary conditions for which it is of great interest to have simple, fast, accurate and reliable numerical algorithms for real time integrations.

It is usual to solve the nonlinear problems by linearization, which leads to problems that are solvable by linear quadratic methods. In general, they require the integration of matrix Riccati differential equations (RDEs) iteratively. The algebraic structure of the RDEs appearing in this problem implies that their solutions are symmetric positive definite matrices, a property that plays an important role for the qualitative and quantitative solutions of both the control and the state vector.

The differential equations to be solved take the form

$$\begin{aligned} \dot{X} &= F(t,X,P), \qquad X(0)=X_0, \quad t\in[0,T] \\ \dot{P} &= R(t,X,P), \qquad P(T)=P_T \end{aligned} \tag{1}$$

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where $X \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$. The equation for P corresponds to a matrix RDE with the solution being a positive definite matrix. We solve this BVP by taking a fixed mesh, $t_i = ih$, $i = 0, 1, \ldots, N$, with $h = T/N$, and using an iterative method

$$P^{n+1} = \phi(X^n, P^n),$$
$$X^{n+1} = \varphi(X^n, P^{n+1}), \tag{2}$$

where

$$X^m = (X_0^m, X_1^m, \ldots, X_N^m), \qquad P^m = (P_0^m, P_1^m, \ldots, P_N^m), \quad m = 0, 1, \ldots$$

with $\lim_{m\to\infty} X_i^m = X_i^{[\infty]} \simeq X(t_i)$, and $X_0^m = X_0, P_N^m = P_T$. Then, we look for a numerical method with the following properties: (i) it only uses values from previous iterations which are evaluated on the fixed mesh, (ii) the algorithms are computationally relatively cheap to evaluate at each iteration (the first iterations can differ considerably from the exact solution and thus involved and computationally costly methods at each iteration are inefficient), (iii) the method has good stability properties in order to allow for relatively large time steps and thereby reduce the storage requirements (X^n, P^n have to be stored), (iv) the method preserves positivity for the approximations of P. Close to convergence, the last iteration can be carried out using a more elaborated and accurate method which can use the values of previous iterations at several mesh points.

The numerical integration of matrix RDEs has been extensively studied as well as high order methods which preserve positivity [5,6] but all these methods, when applied to the iterative algorithm, require the evaluation of previous iteration at points which are not in the mesh and thus do not satisfy property (i), which makes them inappropriate in this context.

Geometric numerical integrators are numerical algorithms which preserve most of the qualitative properties of the exact solution. However, the mentioned positivity of the solution in this problem is a qualitative property which is not unconditionally preserved by most methods, geometric integrators included.

We show that some low order exponential integrators[1] unconditionally preserve this property, and higher order methods preserve it under mild constraints on the time step. We refer to these methods as structure preserving integrators, and they will allow the use of relatively large time steps while showing a high performance for stiff problems or problems which strongly vary along the time evolution.

The aforementioned nonlinearities in the control problems can be dealt with in different ways. We consider three techniques to linearize the equations and the linear equations are then numerically solved using some exponential integrators which preserve the relevant properties of the solution. Since the nonlinear problems are solved by linearization, we first examine the linear problem in detail.

The paper is organized as follows: The linear case is studied in Section 2, where we emphasize the algebraic structure of the equations and the qualitative properties of the solutions. We next consider some exponential integrators and analyze the preservation of the qualitative properties of the solution by the proposed methods. In Section 3, it is shown how the full nonlinear problem can – after linearization – be treated as a particular case of the non-autonomous linear one. The work concludes with the application of the numerical algorithm to a particular example (control of a quadrotor) in Section 4, with which the accuracy of the exponential methods is demonstrated. Numerical results and conclusions are included.

2. Linear quadratic (LQ) methods in optimal control problems

Let us consider the general LQ optimal control problem

$$\min_{u\in L^2} \int_0^{t_f} \left(X^T(t)Q(t)X(t) + u^T(t)R(t)u(t)\right) dt \tag{3a}$$

$$\text{subject to } \dot{X}(t) = A(t)X(t) + B(t)u(t), \qquad X(0) = X_0, \tag{3b}$$

where $\dot{X}(t)$ is the time-derivative of the state vector $X(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the control, $R(t) \in \mathbb{R}^{m\times m}$ is symmetric and positive definite, $Q(t) \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and M^T denotes the transpose of a matrix M.

Problems of the type (3) are frequent in many areas, such as game theory, quantum mechanics, economy, environment problems, etc., see [7,8], or in engineering models [9, Chapter 5].

The optimal control problem (3) is solved, assuming some controllability conditions, by the linear feedback controller [10]

$$u(t) = -K(t)X(t), \tag{4}$$

with the gain matrix

$$K(t) = R^{-1}(t)B^T(t)P(t),$$

[1] We say that a method is an exponential integrator if it uses the computation of matrix exponentials in the algorithm. The same name also refers to a class of numerical integrators used for solving quasilinear differential equations of the form: $\dot{u} = Du + N$, where typically D is a stiff linear operator and N is a slowly varying nonlinear part.