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High order splitting schemes with complex timesteps and their application in mathematical finance

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ABSTRACT

High order splitting schemes with complex timesteps are applied to Kolmogorov backward equations stemming from stochastic differential equations in the Stratonovich form. In the setting of weighted spaces, the necessary analyticity of the split semigroups can easily be proved. A numerical example from interest rate theory, the CIR2 model, is considered. The numerical results are robust for drift-dominated problems and confirm our theoretical results.

1. Introduction

In mathematical finance, the pricing of derivative contracts can be reduced to the calculation of expected values under the risk-neutral measure; see, e.g., [1,2]. This can be performed in different manners. The most general approach is to use Monte Carlo methods [3,4]. Here, the recently introduced multilevel Monte Carlo approach [5] is a fast method for problems with very low smoothness.

In contrast, we want to obtain approximations of the solution of the Kolmogorov backward PDE by splitting schemes. Such methods were also applied successfully in quasi-Monte Carlo simulations in [6], but all such approaches are in the end limited by the accuracy of the integration scheme. Furthermore, it is not straightforward to evaluate the stochastic processes at complex timesteps, which is necessary if splittings of order higher than 2 are to be used.

We therefore solve the PDE by finite element methods in space and a high order splitting method in time. Such high order splitting methods, overcoming the order barrier of 2 commonly encountered for splittings with nonnegative times, see [7], were introduced in [8,9] and require analyticity of the split semigroups. To show this analyticity, we make use of function spaces endowed with weighted supremum norms, originally introduced in [10] for proving the existence of solutions to martingale problems for stochastic partial differential equations and used for the analysis of numerical methods for stochastic partial differential equations in [11–15]. It turns out that using this framework, it is very simple to prove analyticity for semigroups stemming from stochastic differential equations in the Stratonovich form; hence optimal rates of convergence follow. In particular, these results apply to problems on unbounded domains with unbounded coefficients vanishing at the (finite) boundaries of the domain. Such problems are usually difficult to deal with in Sobolev spaces and require the use of weighted Sobolev norms vanishing at the finite boundaries; see [16,17] for some recent results on the Heston stochastic volatility model and a thorough discussion of references on this subject.

A particularly interesting feature of the considered numerical method is that the drift part is completely separated from the diffusion part. This means that we can choose suitable numerical schemes for each of these parts separately. In particular,

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if the first order hyperbolic drift part is solved by a method such as streamline diffusion finite elements or a discontinuous Galerkin method, we can expect that the method is robust for vanishing diffusivity. The need for such methods, notably for applications in mathematical finance, was recently observed in [18]. Other methods yielding such robustness are streamline diffusion methods, see [19,20], and discontinuous Galerkin methods, see, e.g., [21,22]. We stress that an advantage of our method is that different, optimised solvers for the drift part and the diffusion part can be used, providing a more flexible scheme.

The paper is organised as follows. Section 2 recalls the definitions of weighted spaces, specialising the results from [10] and [11,14] to the finite-dimensional case. Furthermore, we show analyticity for a wide class of Markov semigroups in the setting of weighted spaces. Next, Section 3 formulates the splitting scheme and contains a convergence result. Finally, we show numerical findings for the CIR2 model in Section 4. In particular, we observe robustness of the suggested method for drift-dominated problems.

2. The functional analytic framework

Let us start off by recalling some basic facts from stochastic analysis; see, e.g., [23–25] for more details. Fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ and a d-dimensional standard Brownian motion $(W_t^j)_{j=1,\dots,d,t\geq 0}$ on it. Let $N \in \mathbb{N}$. Consider a stochastic differential equation in the Stratonovich form on the closure $D \subset \mathbb{R}^N$ of a (not necessarily bounded) Lipschitz domain,

$$\mathrm{d}X(t,x) = \sum_{j=0}^{d} V_j(X(t,x)) \circ \mathrm{d}W_t^j, \qquad X(0,x) = x, \tag{1}$$

where $W_t^0 = t$ and $\circ\mathrm{d}W_t^0 = \mathrm{d}t$, $V_j : D \to \mathbb{R}^N$ are Lipschitz continuous vector fields with appropriate smoothness assumptions discussed below in more detail, and $x \in D$. We suppose that the solution $(X(t,x))_{t\geq 0}$ is well-defined in D, in particular that up to some common null set $N \in \mathcal{F}$, $\mathbb{P}(N) = 0$, $X(t,x) \in D$ for all $x \in D$ and $t \geq 0$. This is justified by well-known results on stochastic flows; see, e.g., [25, Section V.7]. For $f \in \mathrm{C}^2(D)$ with suitable growth at infinity, Itô's lemma shows that $u(t,x) := \mathbb{E}[f(X(T-t,x))]$ satisfies the backward Kolmogorov equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,x) + \mathcal{L}u(t,x) = 0, \quad t > 0,\ x \in D, \tag{2a}$$

$$u(T,x) = f(x), \quad x \in D. \tag{2b}$$

Here, $\mathcal{L}$ denotes the "sum of squares" partial differential operator, defined for $g \in \mathrm{C}^2(D)$ by

$$\mathcal{L}g(x) := V_0 g(x) + \frac{1}{2}\sum_{j=1}^{d} V_j^2 g(x), \tag{3}$$

with $Vg(x) := V(x) \cdot \nabla g(x)$ the directional derivative for $V : D \to \mathbb{R}^N$ and $g \in \mathrm{C}^1(D)$. We split this operator into

$$\mathcal{L}_0 g(x) := V_0 g(x) \quad \text{and} \quad \mathcal{L}_1 g(x) := \frac{1}{2}\sum_{j=1}^{d} V_j^2 g(x). \tag{4}$$

The respective split stochastic differential equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}X^0(t,x) = V_0(X^0(t,x)), \qquad X^0(0,x) = x, \quad \text{and} \tag{5}$$

$$\mathrm{d}X^1(t,x) = \sum_{j=1}^{d} V_j(X^1(t,x)) \circ \mathrm{d}W_t^j, \qquad X^1(0,x) = x; \tag{6}$$

cf. also [6].

We recall the following definitions from [10,11,14,15].

Definition 1. A function $\psi : D \to (0,\infty)$ is called a *D-admissible weight function* if $\lim_{\|x\|\to\infty, x\in D} \psi(x) = \infty$ and ψ is bounded on compact sets, where $\|\cdot\|$ denotes the Euclidean norm.

Note that the definition from [15] simplifies in this case, as D is locally compact. While the results in [10,11,15] are stated for real-valued functions only, they also hold true for the complex-valued versions of the spaces considered here.

Definition 2. Fix $k \in \mathbb{N}_0$. For $j = 0, \dots, k$, let $\psi_j : D \to (0,\infty)$ be D-admissible weight functions. The space $\mathcal{B}_k^{(\psi_j)_{j=0,\dots,k}}(D)$ is defined as the closure of $\mathrm{C}_b^k(D)$, the space of functions $f : D \to \mathbb{C}$ such that f is bounded and k times differentiable with all derivatives up to order k continuous and bounded, with respect to the norm $\|\cdot\|_{(\psi_j)_{j=0,\dots,k},k}$, where

$$\|f\|_{(\psi_j)_{j=0,\dots,k},k} := \|f\|_{\psi_0} + \sum_{j=1}^{k} |f|_{\psi_j,j} \tag{7}$$