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Semi-analytical methods for singularly perturbed multibody system models

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ABSTRACT

Multibody system models with either small masses or large stiffness terms may cause high computation time due to high frequency oscillations. A method to integrate such problems is motivated by results from singular perturbation theory which relate the solution of the ODE

$$\dot{u} = f(u, v), \qquad \varepsilon \dot{v} = g(u, v)$$

with a small parameter $\varepsilon > 0$ to the solution of the DAE

$$\dot{u}_0 = f(u_0, v_0), \qquad 0 = g(u_0, v_0).$$

For most applications in multibody dynamics, the transformation of the linearly implicit second order model equations to this canonical form is not obvious because of non-diagonal mass matrices, constraints and large stiffness terms in the right hand side.

In the present paper, we consider singularly perturbed second order problems with non-diagonal mass matrices and investigate scaling for large stiffness terms in flexible multibody systems taking into account the structure of second order equations. Furthermore, the approach is generalized to constrained systems. The computational savings of the proposed quasistatic approximation are illustrated by numerical test results for a flexible four bar mechanism.

1. Introduction

The complexity of models in technical simulation may cause substantial problems in time integration. High frequency oscillations with very small amplitudes may be considered as a typical example of model components that are not relevant for practical application but cause stability problems in explicit time integration methods and may slow down implicit integrators because of convergence problems in the corrector iteration [1]. Often, singular perturbation theory allows a detailed analysis of these phenomena [2–4]. For a canonical singularly perturbed problem

$$\dot{x}(t) = f(x(t), y(t)), \qquad x(0) = \bar{x} \tag{1a}$$

$$\varepsilon \dot{y}(t) = g(x(t), y(t)), \qquad y(0) = \bar{y} \tag{1b}$$

with a small parameter $\varepsilon > 0$, we call

$$\dot{x}_0(t) = f(x_0(t), y_0(t)), \qquad x_0(0) = \bar{x}$$

$$0 = g(x_0(t), y_0(t))$$

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the *quasistatic approximation* [5]. If f, g are smooth functions and g_y is invertible, we can solve the quasistatic problem. If additionally the real parts of all eigenvalues $\lambda[g_y]$ are below some ε-independent, negative constant, then there is $\bar{y}(\varepsilon, \bar{x})$ such that the solution of (1) for $\bar{y} = \bar{y}(\varepsilon)$ is smooth w.r.t. t and of the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$$
$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$$

with ε-independent functions x_k, y_k. For initial values $\bar{y} \neq \bar{y}(\varepsilon, \bar{x})$, the solution of (1) differs from the smooth solution by an exponentially decaying term $(\xi, \eta) = (x(t, \varepsilon, \bar{y}), y(t, \varepsilon, \bar{y})) - (x(t, \varepsilon, \bar{y}(\varepsilon, \bar{x})), y(t, \varepsilon, \bar{y}(\varepsilon, \bar{x})))$. This can be summarized in compact form [6] as

Theorem 1. *For the initial value problem (1) with smooth functions f, g suppose that* $\operatorname{Re}\lambda[g_y] \leq -\beta$ *for some positive constant β. If $(\bar{x}, \bar{y})$ lies in a neighborhood of the initial values $(x_0(0), y_0(0))$ of the quasistatic solution, then the problem has a unique solution for $\varepsilon > 0$ sufficiently small and for $t \in [0, t_{\mathrm{end}}]$, which is of the form*

$$x(t) = x_0(t) + \mathcal{O}(\varepsilon)$$
$$y(t) = y_0(t) + \eta_0(t/\varepsilon) + \mathcal{O}(\varepsilon)$$

with $\|\eta_0(\tau)\| \leq \|\eta_0(0)\| e^{-\beta\tau}$.

The main interest of the present paper is in linearly implicit second order systems that represent the equations of motion for multibody systems. They are in the form

$$M\ddot{q} = f(q, \dot{q}), \tag{2}$$

where q denotes the vector of position, orientation and elastic deformation of all bodies [7] and f is the force vector. The symmetric, positive definite mass matrix M includes all masses and inertia terms. The Jacobian $f_{\dot{q}}$ characterizes linearized damping terms and $-f_{\dot{q}}$ is referred to as damping matrix. If we can separate the singularly perturbed terms in (2) by splitting $M = M_0 + \varepsilon M_1$ with $\operatorname{rank} M > \operatorname{rank} M_0$ and M_1 positive semidefinite, we call

$$M_0\ddot{q}_0 = f(q_0, \dot{q}_0) \tag{3}$$

the (linearly implicit) quasistatic approximation of the second order ODE (2). The benefits of this approach were studied for rigid body systems without constraints in a previous work [8].

In general, the singular perturbation cannot be found straightforwardly by looking at the equations of motion for multibody systems. For multibody systems, singular perturbations may result from small inertia terms, strong damping forces [9] and large stiffness terms [1]. In the constrained case, the complex coupling structure may eliminate implicitly the singularly perturbed terms. Moreover, simply neglecting small mass matrix entries could yield an indefinite matrix due to coupling terms which violates physical laws since $M = \frac{\partial^2}{\partial \dot{q}^2} T$ where the kinetic energy T is a positive semidefinite function of $\dot{q}$, see [10] and Section 4.

In the present paper, we describe a technique to reduce the computation time in multibody system simulation using results from perturbation theory. A significant speed-up is gained for both the second order ODE (2) and the equations of motion with constraints for complex multibody system models.

The basic idea of the quasistatic approximation of equations of motion is demonstrated for a walking mobile robot model in Section 2, see also [8]. In the next section, we construct a general method to adjust a pre-selected block in a non-diagonal matrix to gain a quasistatic approximation. Application of this method to positive definite matrices will result in a problem with positive semidefinite matrix. Section 4 is devoted to model flexible multibody systems which illustrate the typical structure of equations of motion in multibody dynamics with large stiffness terms. We investigate scaling flexible multibody systems taking into account the structure of second order equations. The quasistatic approximation for multibody system models with constraints is considered in Section 5. Finally, we present in Section 6 the savings and limitations of the quasistatic approximation for a flexible four bar mechanism.

2. Motivation

As an introducing example, we consider a walking mobile robot model that was proposed in [11]. For sake of simplicity, only a planar mobile robot was described in [8], see Fig. 1. It consists of a rectangular platform (mass $m = 1$, moment of inertia $I = 1$) and two legs that are controlled by two telescopic actuators. The position of the platform is given by the Cartesian coordinates x_S, y_S of center S of mass and the angular coordinate φ_S describing the rotation of the platform around S. We assume that the mass $m_N = 0.02$ of leg j is concentrated in contact point $N_j = (x_{Nj}, y_{Nj})$. To avoid unilateral constraints, the contact between leg and ground is modeled by springs.

The walk of the mobile robot consists of several maneuvers which are executed by changing the length L_{ij} of the actuators R_iN_j, see [11]. With PID controllers, the force between R_i and N_j becomes

$$F_{ij} = k_P(L_{ij} - L_{ij}^{\mathrm{d}}) + k_I \int_0^t (L_{ij} - L_{ij}^{\mathrm{d}})\mathrm{d}\tau + k_D \frac{\mathrm{d}}{\mathrm{d}t} L_{ij},$$