Journal of Computational and Applied Mathematics 262 (2014) 373–383

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Mixed-mode oscillations and chaotic solutions of *jerk* (Newtonian) equations[☆]

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ARTICLE INFO

Article history:
Received 25 October 2012
Received in revised form 29 May 2013

Keywords:
Mixed-mode oscillations
Jerk equations
Newtonian oscillations
Bifurcations
Chaos

ABSTRACT

We analyze *jerk* equations (third-order ODEs) resulting from an underlying prototypical model of mixed-mode oscillations and propose their circuit realizations in this paper. The scalar ODEs and their corresponding circuit realizations are obtained from a system of first-order ODEs with one nonlinearity (third-degree polynomial). One of the *jerk* equations is Newtonian as it is obtained by computing the time-derivative of the second Newton's law $x'' - F/m = 0$ for a constant mass m and specially designed nonlinear force function $F(x, x', \tau)$. The second *jerk* equation is non-Newtonian. The two circuits are op-amp RC circuits with interesting dynamical properties, including the mixed-mode and chaotic oscillations. The mixed-mode oscillations follow the rules of Farey arithmetic and the circuits' dynamics is of a fractal nature.

1. Introduction

Many third-order nonlinear dynamical systems show interesting properties, time domain responses and bifurcation patterns. The specific features of such systems are their oscillatory responses (including chaotic ones), periodic mixed-mode sequences, existence of various types of attractors, canard solutions and often surprising bifurcation diagrams when one or more parameters vary. The famous Lorenz, Rössler and Chua systems can be mentioned as classical examples of such systems [1,2]. Interesting features of many three-variable nonlinear systems have been reported not only in electrical and mechanical engineering, biological processes and chemical reactions, but also in economics, plasmas, cardiac dysrhythmias of human heart and synchronization of neurons in human brains [3,4]. Important properties of most of those systems, including the singularly perturbed ones, are mixed-mode oscillations (or MMOs). MMOs are complex responses with both the large and small amplitude oscillations (LAOs and SAOs, respectively). Historically, the Belousov–Zhabotinsky chemical reaction was the first process with MMOs to be reported and examined [5]. Other nonlinear systems and processes with MMOs are discussed in [6–10]. Another important property of many of those systems is their *jerky* nature, when it is possible to transform (at least in some variables) a system of three first-order nonlinear ODEs to the following third-order scalar differential equation

$$X''' = J(X, X', X''), \tag{1}$$

where the prime $'$ denotes the time derivative of $X(t)$. Since the third derivative of the position variable $X(t)$ in mechanics is called the *jerk*, the above equation is called a *jerk* equation for some nonlinear J, the *jerk* function. This terminology has also been used in the areas of electronic circuits and nonlinear processes [11–16].

☆ This paper is based on a talk presented at NUMDIFF-13, Sept. 10–14, 2012 in Halle (Germany).

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0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.07.022

In this paper we extend the discussion on the singularly perturbed three-variable prototypical model of MMOs [17–20]. In particular, we transform the system of three nonlinear ODEs in that model to two equivalent *jerk* equation (1). We show that one of the *jerk* equations is Newtonian as it can be linked to the second Newton's law. Through various bifurcation diagrams and time-series solutions of the Newtonian *jerk* circuit we examine its MMOs in the context of Farey arithmetic. Next, the circuit's fractal nature is discussed and the attractor's capacity dimension estimated to be about 2.4. A specific property of the circuit, saturation phenomenon of the circuit's operational amplifiers (op-amps), is also examined using Spice and Matlab software.

The main purpose of writing the paper is to analyze the two *jerk* equations with their circuit implementations and to examine a rather wide spectrum of their dynamical properties, ranging from MMOs, through chaotic responses, various bifurcation diagrams related to the Farey sequence of coprime integers, to fractal properties. A link of one of the *jerk* equations to the second Newton's law allows for a *mechanical* interpretation of the electrical variables of the circuit as position, velocity and acceleration.

To put the topic of *jerk* in a perspective we would like to offer the following broader remarks. Studying *jerk* has become an important issue in many areas, particularly in engineering, biology, chemistry and even every day life of a human being. Roller coasters are designed to limit the *jerk* on human bodies and to allow time to sense stress changes and adjust muscle tension or suffer a whiplash. In manufacturing processes, rapid changes in acceleration of a cutting tool may cause damage of that tool or even completely destroy the cutting process. For this reason, many modern motion controllers include jerk limitation devices. In electric networks, a high inrush current may disturb the networks' dynamics or even damage them to a great extent. A series of rocket sled deceleration experiments a few decades ago showed that a specially trained person survived a peak force of 46.2 times the force of gravity [21]. Studies of ultrashort pulses of light (in the range of femtoseconds to picoseconds) led to the 1999 Nobel Prize in chemistry awarded for using such ultrashort pulses to observe chemical reactions [22].

2. The *jerk* equations

Consider the following three-scale system introduced and analyzed analytically in [17]

$$\begin{aligned} \epsilon x_1' &= -x_2 + \alpha x_1^2 + \beta x_1^3 \\ x_2' &= x_1 - x_3 \\ x_3' &= a - bx_2 \end{aligned} \tag{2}$$

where $0 < \epsilon \ll 1$ is a "small" parameter. The $a = \epsilon\bar{a} > 0$ and $b = \epsilon\bar{b} > 0$ for real numbers $\bar{a}$ and $\bar{b}$. We also have $\alpha > 0$ and $\beta < 0$. The system is called a three-scale one with x_1 and x_3 being the fastest and slowest variables, respectively. System (2) is a prototypical system for MMOs when LAOs and SAOs intermingle yielding interesting L^s responses with L and s denoting the number of large and small oscillations in a period, respectively. Typical MMOs (time-domain and 3D plots) obtained from (2) are shown in Fig. 1 for $L = 3$ and $s = 8$.

The LAOs are simply of relaxation type when, due to a "small" ϵ value, the system approximately traces the branches of $x_2 = \alpha x_1^2 + \beta x_1^3 \equiv N(x_1)$ (the first equation in (2) for $\epsilon = 0$) with negative slopes, including the neighborhoods of the maximum $(-2\alpha/(3\beta), 4\alpha^3/(27\beta^2))$ and minimum $(0, 0)$ points of $N(x_1)$. The SAOs are a result of 2D-like oscillations in (x_1, x_2), see the graph in Fig. 1(b) projected into (x_1, x_2). In 3D the SAOs occur in the fold area of the (x_1, x_2, x_3) space around the origin $(0, 0, 0)$. A series of SAOs around $(0, 0, 0)$ is followed by one or more LAOs (relaxations). The system leaves the fold due to the *canard explosion* phenomenon to enter the relaxation mode. Then, the *return mechanism* after the relaxation phase, allows the system to return back to a neighborhood of the fold for a new series of SAOs in the next period. For example, in one period shown in Fig. 1(b) we have a series of 8 SAOs around the point $(0, 0, 0)$ followed by 3 LAOs when the system follows two negative-slope branches of $N(x_1)$ with two (almost instantaneous) jumps between these branches. It was shown in [17] that the MMOs L^s happen if $a < b\alpha^3/(18\beta^2)$. With the inequality sign reversed we only have LAOs and no SAOs. Special Stern–Brocot trees for (2) are constructed and discussed in [18], while analytical properties are considered in [19]. Finally, a numerical treatment of (2) is proposed in [23], where a Fortran-90 code to solve the three-variable system, compute its bifurcation diagrams and Arnold's tongues is described. The code is applicable for parallel computations.

In this paper we transform (2) into two scalar third order nonlinear equations of the form (1) for variables x_1 and x_3. In mechanics, when X is a position, the nonlinear function J is called a *jerk* since it equals the first derivative of acceleration $x''(t)$. The *jerk* properties of variables x_1 and x_3 in (2) can be examined in the following way.

Theorem 1. *Let $\tau = \gamma t$ with $\gamma > 1$. Variables x_1 and x_3 in (2) satisfy the* jerk *equations*

$$\epsilon\gamma^3 x_1''' - \gamma^2 \dot{N}(x_1)x_1'' + [-\gamma^2 \ddot{N}(x_1)x_1' + \gamma(1-b)]x_1' - a + bN(x_1) = 0 \tag{3}$$

and

$$\epsilon\gamma^3 x_3''' + \gamma(1-\epsilon b)x_3' - a + bN((-\gamma^2 x_3'' + bx_3)/b) = 0, \tag{4}$$

respectively, with $N(x_1) \equiv \alpha x_1^2 + \beta x_1^3$, $\dot{N}(x_1) = d(N(x_1))/dx_1$, $\ddot{N}(x_1) = d^2(N(x_1))/dx_1^2$ in (3) and the corresponding form of $N(.)$ in (4). The $'$ denotes $d/d\tau$.