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A tabular methodology to identify minimal row degrees for Matrix Padé Approximants[☆]

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ABSTRACT

In this paper we propose a tabular procedure for identifying sets of minimal row degrees corresponding to Matrix Padé Approximants. The objective is to make it easier to interpret and properly apply them in various fields. When considering formal matrix power series with $k \times m$ coefficients, it is useful to examine the linearly dependent rows that are in the last k rows in certain Hankel matrices corresponding to the coefficients of the series. A list of properties and suitable examples are added not only by way of illustration but also because they are an original part of the method, having been chosen to teach the procedure and to underscore certain precautions to be taken.

1. Introduction

A canonical Matrix Padé Approximants (MPA) with minimal row degrees is defined and studied in [1], inspired by the original way in which Tiao and Tsay in [2] resolved the identification of time series VARMA models, through what they define as Scalar Component Models (SCM). In fact, the objective of [1] was: to find rational approximations with matrix polynomials of a matrix power series, under certain conditions of minimality on the degrees of the rows of these polynomials, with the constant term in the denominator being invertible and the proposed representation having no free parameters (i.e. is canonical). However, given a set of minimal row degrees (m.r.d.) for an approximant, we propose in [1, Section 5] a unique canonical representation, but an approximant could have several sets of m.r.d. Therefore, we propose in [1] alternative canonical representations for an approximant. In addition, given a set of m.r.d. for an approximant we could define other canonical representations different from those we have defined in [1]. Depending on the field to be used, some of them may be better than others.

This work addresses an open question in [1]: automatically finding all of the possible sets of minimal row degrees of a MPA, for those users who are not familiar with the more mathematical aspects of the results and proofs in [1]. A semi-automatic procedure is proposed in this paper that makes it easier to identify the sets of minimal row degrees to be applied in several fields.

We highlight the difficulties involved in finding the proper uniqueness and minimality of a MPA and the guidance provided by studying these concepts using a tabular methodology. By way of examples related to the process used in this paper, in the scalar case, Table C provides a fundamental tool by characterizing a rational function and the degrees of the

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polynomials that represent it in its unique irreducible or minimal form. In the matrix case, the concepts of rationality, minimality and uniqueness take on a greater complexity and independence. For example, the Hankel determinants shown in Table C lose importance because, among other reasons, they only make sense when the matrices are square. But not only because of this; even if the matrices are square, Table C does not present a structure that allows for a characterization of rationality (see [3]). Then, [4] provides an answer to this question by using the rank of Hankel matrices. Furthermore, given a matrix rational function, the "smallest" degrees of the matrix polynomials which represent it are not necessarily unique. Various types have been defined in the literature involving the degrees of the polynomials that appear in a rational function. For instance, [4] studies a minimal overall degrees, [1] a type of row degrees, etc. In addition, given that the rational representation of the function for the same pair of minimal overall degrees need not be unique, [4] gives conditions to study the uniqueness of said representation. All the results obtained are presented graphically in tables.

This paper is divided into sections as follows. Section 2 outlines the preliminary definitions and notation. Section 3 comprising the bulk of the paper, the tabular methodology to address the question of what sets of row degrees are minimal for the approximants of interest. We illustrate the last step of the tabular procedure with a list of properties and suitable examples chosen to make the methodology easier to understand and apply. Section 4 includes computational aspects of the Mathematica procedure developed to study the examples in the previous section. Section 5 introduces a possible application of this paper. Finally, we present the conclusions, some open questions and the references.

2. Preliminary definitions and notation

Next we summarize the definitions and notation necessary for our contribution, namely, the practical interpretation of the main results in [1], with the help of a suitable tabular methodology.

We will rely on the MPA notation that is most commonly used in the literature.

Definition 1. Starting with $F(z) = \sum_{i=0}^{\infty} c_i z^i$, $c_i \in C^{k\times m}$, $z \in C$, assuming that there exist $P(z)$ and $Q(z)$, of degrees p and q respectively, and of suitable dimensions such that $P(z)F(z) - Q(z) = O(z^{p+q+1})$, then $(P(z), Q(z))$ is said to be *a left MPA of degrees* (p, q), which we shall denote ${}^{L}[q/p]_F$. If $P(z)$ is invertible, then equivalently if $F(z) - P^{-1}(z)Q(z) = O(z^{p+q+1})$, it is said that $P^{-1}(z)Q(z)$ is a *left Padé MPA of degrees* (p, q).

Analogously, a *right MPA of degrees* (p, q) is defined, which we shall denote ${}^{R}[q/p]_F$. We will only deal with the left MPA, though similar results could be obtained for the right MPA.

The polynomial $P(z) = P_0 + P_1 z + \cdots + P_p z^p$ is called the *denominator* and $Q(z) = Q_0 + Q_1 z + \cdots + Q_q z^q$ the *numerator* of the corresponding approximant.

As a consequence of the definition, ${}^{L}[q/p]_F$ exists if and only if the homogeneous system

$$(P_p P_{p-1} \ldots P_0) M(p, q, p+q) = 0 \tag{1}$$

has a solution, where the Hankel matrix $\boldsymbol{M(i, j, h)} = \begin{pmatrix} c_{j-i+1} & c_{j-i+2} & \cdot & \cdot & c_{h-i} \\ c_{j-i+2} & c_{j-i+3} & \cdot & \cdot & c_{h-i+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_j & c_{j+1} & \cdot & \cdot & c_{h-1} \\ c_{j+1} & c_{j+2} & \cdot & \cdot & c_h \end{pmatrix}$ for $i, j \in N$, $i \geq 0$, $j \geq 0$ and $h \geq i + j$.

By convention, $c_n = 0 \in C^{k\times m}$ if $n < 0$ and $M(0, j, j) = 0 \in C^{k\times m}$. In particular, ${}^{L}[q/p]_F$ with P_0 invertible exists if and only if (1) has a solution with P_0 invertible, which is equivalent to $\operatorname{rank} M(p-1, q-1, p+q-1) = \operatorname{rank} M(p, q, p+q)$.

It is a trivial observation that any approximant of degrees (p, q), $P^{-1}(z)Q(z)$, with $P_0 \neq I$ (I the *identity* matrix) but invertible, has another representation, $(P_0^{-1}P(z))^{-1}(P_0^{-1}Q(z))$, with the same degrees (p, q) but with constant term $P_0^{-1}P(0) = I$ in the denominator.

The main concepts in this paper are in the following definitions.

Definition 2 (*[4]*)**.** Let $A(z) = P^{-1}(z)Q(z)$ a matrix rational function and $P_0 = I$. The degrees p and q of the matrix polynomials $P(z)$ and $Q(z)$, are considered to be *minimal overall (to the left)* if and only if for any two other polynomials $D(z)$ and $N(z)$ of degrees d and n respectively, which satisfy $D(0) = I$ and $A(z) = D^{-1}(z)N(z)$, it holds that $n < q$ implies $d > p$ and $d < p$ implies $n > q$.

Definition 3 (*[1]*)**.** $A(z)$, a matrix rational function of dimension $k \times m$, has at least one *row representation* (a, b), if there exist $P(z) = P_0 + P_1 z + \cdots + P_p z^p$ and $Q(z) = Q_0 + Q_1 z + \cdots + Q_q z^q$, matrix polynomials of dimensions $k \times k$ and $k \times m$, respectively, P_0 being invertible, $A(z) = P^{-1}(z)Q(z)$ and there exists $i \in \{1, 2, \ldots, k\}$ such that the degree of the i-th row of $P(z)$ is a and the degree of the i-th row of $Q(z)$ is b.

Moreover, we say that: (a, b) is a pair of *row degrees*, the set C with the k pairs of row degrees is a *set of row degrees* for $A(z)$ and we use $\mu_C(a, b)$ to denote the times that (a, b) is repeated in C.

In general, a row representation is not associated with a specific row (see, for instance Example 3.1 in [1]).