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New efficient estimators in rare event simulation with heavy tails

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ABSTRACT

This paper is concerned with the efficient simulation of $\mathbb{P}(S_n > s)$ in situations where s is large and S_n is the sum of n i.i.d. heavy-tailed random variables $X_1, \ldots, X_n$. The most efficient and simplest estimators introduced in the rare event simulation literature are those proposed by Asmussen and Kroese (2006) and Asmussen and Kortschak (2012). Although the main techniques for facing the rare event problem are importance sampling and splitting, the estimators of Asmussen, Kortschak and Kroese combine exchangeability arguments with conditional Monte-Carlo to construct estimators whose relative errors go to 0 as $s \to \infty$. In this paper, we decompose $\mathbb{P}(S_n > s)$ as the sum of $\mathbb{P}(M_n > s)$ and $\mathbb{P}(S_n > s, M_n < s)$ as proposed by Juneja (2007) because $\mathbb{P}(M_n > s)$ is known in closed form and is asymptotically equivalent to $\mathbb{P}(S_n > s)$. We construct new efficient estimators of $\mathbb{P}(S_n > s, M_n < s)$ by splitting up it again and then by using the same type of reliable methods as in Asmussen and Kroese (2006). We show that these new estimators have smaller relative errors than the estimators of Asmussen, Kortschak and Kroese. The conclusion of the numerical study is that our estimators compare extremely favorably with previous ones.

1. Introduction

In a probabilistic model, a rare event is an event with a very small probability of occurrence (typically between 10^{-6} and 10^{-10} or less). Rare events are of particular practical interest when dealing with systems where the rare event is a catastrophic failure with possible important human or monetary losses. Typical examples are given by failures in a public transport system or in a nuclear power plant, failures of information processing systems or telecommunication networks, ruins of a large number of insurance companies or banks, and so on. Therefore it is a critical issue to be able to evaluate the probability of these rare events.

In many cases, the mathematical models are often too complicated to be solved by analytical or numerical methods. Soft computing models and techniques are often used for complex systems that remain intractable to conventional mathematical and analytical methods. Among others we can quote artificial neural network models, image processing techniques, data-driven models coupled with data-preprocessing techniques, classifier ensemble methods, artificial neural network simulation methods (see e.g. for real applications: [1–6]).

The simulation of the probabilistic models and the use of Monte Carlo methods provide more interesting alternative tools. However, the estimation of rare event probabilities with the naive Monte Carlo techniques requires a prohibitively large number of trials. It is therefore necessary to use suitable variance reduction techniques as control variates, antithetic sampling, importance sampling, splitting, and so on. In this paper, we focus on techniques for the efficient estimation of tail probabilities involving sums of heavy tailed random variables. It is amongst the simplest problems studied in the literature

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of rare event simulation, but also amongst the most challenging (see e.g. Chapter VI in [7]). Such probabilities are of interest for insurance companies: the wealth of the company is modeled as a stochastic process that incorporates the gains due to insurance premiums and the losses due to claims. The failure event corresponds to the ruin of the company, i.e. when its wealth becomes negative.

More specifically, we consider non-negative, independent, identically distributed (i.i.d.) regularly varying heavy-tailed random variables $X_1, \ldots, X_n$ with common distribution F and let $S_n = X_1 + \cdots + X_n$. We are interested in the problem of efficiently estimating

$$z(s) = \mathbb{P}(S_n > s)$$

for large s. It is well-known that rare event simulation techniques are quite different in the light-tailed setting and in the heavy-tailed setting (see e.g. Chapter VI in [7]). The simulation method we propose here is only concerned with regularly varying heavy-tailed distributions which are one of the most important subclasses of heavy-tailed distributions. However modifications of the algorithm would be needed for other classes of distributions.

Let us denote by $Z = Z(s)$ a random variable that can be generated by simulation and has expectation equal to $z(s)$. The usual performance measure is the relative error $e(s) = (\mathbb{V}\mathrm{ar}(Z))^{1/2}/z(s)$. An estimator has the logarithmically efficient property if $\limsup_{s\to\infty} z(s)^{\epsilon} e(s) < \infty$ for all $\varepsilon > 0$, it has the bounded relative error property if $\limsup_{s\to\infty} e(s) < \infty$ and it has the vanishing relative error property if $\lim_{s\to\infty} e(s) = 0$.

The first logarithmically efficient estimator was proposed by [8] and was based on a conditioning method. Asmussen et al. [9] and Juneja and Shahabuddin [10] used importance sampling techniques for estimating $z(s)$ and obtained logarithmically efficient estimators. Asmussen and Kroese [11] proposed an estimator that combines an exchangeability argument with a conditional Monte Carlo idea

$$Z_{AK} = n\bar{F}\left(M_{n-1} \vee (s - S_{n-1})\right)$$

where $M_{n-1} = \max(X_1, \ldots, X_{n-1})$ and $S_{n-1} = X_1 + \cdots + X_{n-1}$. This estimator is shown in [11] to have bounded relative error and in [12] to have vanishing relative error. The exact rates of decay of $e(s)$ for Z_{AK} have been recently given in [13] under the assumption that the probability density function of X_1 exists and is given by $f(x) = \alpha L(x)/x^{\alpha+1}$ where L is a slowly varying function. If $\alpha > 2$, they showed that, as $s \to \infty$,

$$e_{AK}(s) = (1 + o(1))\,\alpha\left[(n-1)\,\mathbb{V}\mathrm{ar}(X_1)\right]^{1/2} s^{-1},$$

and if $\alpha < 2$, that, as $s \to \infty$,

$$e_{AK}(s) = (1 + o(1))\left[(n-1)\,k_\alpha \bar{F}(s)\right]^{1/2},$$

where k_α is a positive constant that depends on α (see [13] for the case $\alpha = 2$).

Asmussen and Kortschak [13] also introduced related estimators with faster rates of decay for the two cases: $\alpha > 2$ and $1 < \alpha < 2$. If $\alpha > 2$, they suggested the following estimator:

$$Z_{AKo^{(1)}} = Z_{AK} + n\left(\mathbb{E}[S_{n-1}] - S_{n-1}\right) f(s). \tag{1.1}$$

They gave its exact rates of decays when the first derivative of f exists and is given by $f'(x) = -\alpha(\alpha-1)L(x)/x^{\alpha+2}$. They showed that, if $\alpha > 4$, then

$$e_{AKo^{(1)}}(s) = (1 + o(1))\,\frac{\alpha(\alpha-1)}{2}\left[\mathbb{V}\mathrm{ar}\left(S_{n-1}^2\right)\right]^{1/2} s^{-2},$$

as $s \to \infty$, and if $2 < \alpha < 4$, then

$$e_{AKo^{(1)}}(s) = (1 + o(1))\left[(n-1)\,k'_\alpha \bar{F}(s)\right]^{1/2},$$

as $s \to \infty$, where k'_α is a positive constant that depends on α (see [13] for the case $\alpha = 4$). If $1 < \alpha < 2$, they suggested to use an importance sampling method to improve on Z_{AK} and proposed to consider the estimator $Z_{AKo}^{(2)} = Z^{(b)} + nZ^{(c)}$ where

$$Z^{(b)} = n(n-1)\,\bar{F}\left(s/(2(n-1))\right)^2 I\left(S_n > s, X_{n-1} \wedge X_n \geq M_{n-2}\right)$$
$$Z^{(c)} = \left(\bar{F}(s - S_{n-1}) - \bar{F}(s)\right) I\left(M_{n-1} \leq \frac{s}{2(n-1)}\right) R + \bar{F}(s)\mathbb{P}\left(M_{n-1} \leq \frac{s}{2(n-1)}\right)$$

with $R = \prod_{i=1}^{n-1} f(X_i)/\tilde{f}(X_i)$ and $\tilde{f}$ an importance sampling density of the form $\tilde{L}(s)/s^{\tilde{\alpha}}$ such that $\tilde{\alpha} < 2\alpha - 2$. They showed that, as $s \to \infty$,

$$e_{AKo^{(2)}}(s) = O\left(s^{-1}\right).$$

Ghamami and Ross [14] used the first time when the sum of the current maximum and S_n exceeds s to improve Z_{AK}: numerical examples for the standard Weibull distribution show that a variance reduction is gained despite a four time longer computer time, but no theoretical result on the rate of decay of the relative error is given. Other algorithms that share the feature of vanishing relative error have also been investigated but they could be more complicated to implement