Journal of Computational and Applied Mathematics 261 (2014) 85–94

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the singularity of multivariate Hermite interpolation[☆]

Zhaoliang Meng[a,*], Zhongxuan Luo[a,b]

[a] School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China

[b] School of Software, Dalian University of Technology, Dalian, 116620, China

ARTICLE INFO

Article history:
Received 10 June 2013
Received in revised form 24 October 2013

Keywords:
Hermite interpolation
Singularity
Interpolation space
Polynomial ideal

ABSTRACT

In this paper, we study the singularity of multivariate Hermite interpolation of type total degree. We present two methods to judge the singularity of the interpolation schemes considered and by methods to be developed, we show that all Hermite interpolation of type total degree on $m = d+k$ points in $\mathbb{R}^d$ is singular if $d \geq 2k$. And then we solve the Hermite interpolation problem on $m \leq d+3$ nodes completely. Precisely, all Hermite interpolations of type total degree on $m \leq d+1$ points with $d \geq 2$ are singular; only three cases for $m = d+2$ and one case for $m = d+3$ can produce regular Hermite interpolation schemes, respectively. Besides, we also present a method to compute the interpolation space for Hermite interpolation of type total degree.

1. Introduction

Let Π^d be the space of all polynomials in d variables, and let Π^d_n be the subspace of polynomials of total degree at most n. Let $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ be a set of pairwise distinct points in $\mathbb{R}^d$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_m\}$ be a set of m nonnegative integers. The Hermite interpolation problem to be considered in this paper is described as follows: find a (unique) polynomial $f \in \Pi^d_n$ satisfying

$$\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_d}}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}} f(X_q) = c_{q,\boldsymbol{\alpha}}, \quad 1 \leq q \leq m,\ 0 \leq |\boldsymbol{\alpha}| \leq p_q, \tag{1}$$

for given values $c_{q,\boldsymbol{\alpha}}$, where the numbers p_q and n are assumed to satisfy

$$\binom{n+d}{d} = \sum_{q=1}^{m} \binom{p_q+d}{d}. \tag{2}$$

Following [1,2], such a kind of problem is called Hermite interpolation of type total degree. The interpolation problem $(\mathbf{p}, \mathscr{X})$ is called regular if the above equation has a unique solution for each choice of values $\{c_{q,\boldsymbol{\alpha}}, 1 \leq q \leq m, 0 \leq |\boldsymbol{\alpha}| \leq p_q\}$. Otherwise, the interpolation problem is singular. As shown in [3], the regularity of Hermite interpolation problem $(\mathbf{p}, \mathscr{X})$ implies that it is regular for almost all $\mathscr{X} \subset \mathbb{R}^d$ with $|\mathscr{X}| = m$.

Definition 1 (*[3]*)**.** We say that the interpolation scheme $\mathbf{p}$ is:

- regular if the problem $(\mathbf{p}, \mathscr{X})$ is regular for all $\mathscr{X}$.
- almost regular if the problem $(\mathbf{p}, \mathscr{X})$ is regular for almost all $\mathscr{X}$.
- singular if $(\mathbf{p}, \mathscr{X})$ is singular for all $\mathscr{X}$.

☆ This project is supported by NNSFC (Nos. 11301053, 61033012, 11171052, 11271060, 61272371) and "the Fundamental Research Funds for the Central Universities".

* Corresponding author. Tel.: +86 13624949241.
E-mail address: mzhl@dlut.edu.cn (Z. Meng).

0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.10.049

The special case in which the p_q are all the same is called the uniform Hermite interpolation of type total degree. In the case of uniform Hermite interpolation of type total degree, Eq. (2) should be changed to

$$\binom{n+d}{d} = m\binom{p+d}{d}. \tag{3}$$

The research of regularity of multivariate Hermite interpolation is more difficult than the Lagrange case, although the latter is also difficult. One of the main reasons is that Eq. (2) or (3) does not hold in some cases. Up to now, we have known that all the Hermite interpolation on $m \le d+1$ points are singular except for the Lagrange interpolation, see [1,2,4]. Besides, no other results appeared for $m \ge d+2$. Actually, the Hermite interpolation of type total degree on $d+2$ nodes in $\mathbb{R}^d$ is not necessarily singular. For more research of this area, we can refer the reader to [5,2,1,6,4,7–12] and the reference therein.

The main purpose of this paper is to investigate the singularity of Hermite interpolation for $m = d+k$ with $k = 1, 2, 3$. This paper will propose two ways to prove the singularity of Hermite interpolation of type total degree. The first one consists of constructing the interpolation space from the view of polynomial ideal. We state this method in a general way which is also useful for other types of interpolations. The second one depends on an algorithm (see Theorem 5) from which we can get a polynomial being a solution of the homogeneous interpolation problem. This method leads to the most general singularity theorems. To get complete results for $m = d+k$ with $k = 1, 2, 3$, we employ the second method to get the results for general cases and employ the first one for other special cases.

By the presented methods, we show that all Hermite interpolation of type total degree on $m \le d+1$ nodes in $\mathbb{R}^d$ are singular except for the Lagrange interpolation; on $m = d+2$ nodes in $\mathbb{R}^d$ are singular except for three cases; on $m = d+3$ nodes are singular except for one case. Moreover, we also show all the Hermite interpolation problems of type total degree with $m = d+k$ nodes are singular for $d \ge 2k$. The result of $m \le d+1$ is well known, but our method is different. To the best of our knowledge, all the results except for the case of $m \le d+1$ seem to be new.

This paper is organized as follows. In Section 2, we will consider the interpolation space satisfying the Hermite interpolation requirement from the view of polynomial ideal. In this section, Eq. (2) is not required and the polynomial space is not necessary Π_n^d. In Section 3, we consider the singularity of the Hermite interpolation of type total degree and present the main results. Finally, in Section 4, we conclude our results.

2. Interpolation space

It is well known that polynomial interpolation is closely related to the polynomial ideal. This relation is implied in many early papers and is widely employed, for example [13–15]. In [16], Xu presented a solution to the Lagrange interpolation problem from the view of polynomial ideal. This section will generalize Xu's results to Hermite case. That is, we will consider the construction of the interpolation space with respect to Hermite interpolation problem. This also proposes an approach to judge the singularity of Hermite interpolation problem of total degree for given $\mathscr{X}$ and $\mathbf{p}$.

Precisely, in this section, we consider the following interpolation problem:

let $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ be a set of pairwise distinct points in $\mathbb{R}^d$ and $\mathbf{p} = \{p_1, p_2, \ldots, p_m\}$ be a set of m nonnegative integers. Find a subspace $\mathscr{P} \subset \Pi^d$ such that for any given real numbers $c_{q,\boldsymbol{\alpha}}$, $1 \le q \le m$, $1 \le |\boldsymbol{\alpha}| \le p_q$ there exists a unique polynomial $f \in \mathscr{P}$ satisfying the interpolation conditions

$$\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_d}}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}} f(X_q) = c_{q,\boldsymbol{\alpha}}, \quad 1 \le q \le m,\ 0 \le |\boldsymbol{\alpha}| \le p_q \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$ and $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

Following [16], we call such a pair $\{\mathscr{X}, \mathbf{p}, \mathscr{P}\}$ correct. Clearly, such a kind of space $\mathscr{P}$ always exists if no any constraint is added. For the Lagrange case, such a kind of interpolation problem was studied extensively by many authors, for example [11,9,10,7] and the reference therein.

Before proceeding, we first present some necessary notations. Throughout of this paper, we use the usual multi-index notation. To order the monomials in $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$, we use the graded lexicographic order. Let I be a polynomial ideal in Π^d. The codimension of I is denoted by codim I, that is,

$$\operatorname{codim} I = \dim \Pi^d / I.$$

If there are polynomials $f_1, f_2, \ldots, f_r$ such that every $f \in I$ can be written as

$$f = a_1 f_1 + a_2 f_2 + \cdots + a_r f_r, \quad a_j \in \Pi^d,$$

we say that I is generated by the basis $f_1, f_2, \ldots, f_r$, and we write $I = \langle f_1, f_2, \ldots, f_r \rangle$.

For a fixed monomial order, we denote by $LT(f)$ the leading monomial term for any polynomial $f \in \Pi^d$; that is, if $f = \sum c_\alpha X^\alpha$, then $LT(f) = c^\beta X^\beta$, where X^β is the leading monomial among all monomials appearing in f. For an ideal I in Π^d other than $\{0\}$, we denote by $LT(I)$ the leading terms of I, that is,

$$LT(I) = \{cX^\alpha \mid \text{there exists } f \in I \text{ with } LT(f) = cX^\alpha\}.$$

We further denote by $\langle LT(I) \rangle$ the ideal generated by the leading terms of $LT(f)$ for all $f \in I \setminus \{0\}$.

The following theorem is important for our purpose.