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A robust overlapping Schwarz domain decomposition algorithm for time-dependent singularly perturbed reaction–diffusion problems

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ABSTRACT

In this work, we consider a singularly perturbed parabolic problem of reaction–diffusion type. To solve this problem numerically we develop an overlapping Schwarz domain decomposition method, where we use the asymptotic behaviour of the exact solution for domain partitioning. We prove that the method gives uniform numerical approximations of first order in time and almost second order in space. Furthermore, we address the much faster convergence of the algorithm for small perturbation parameter ε. To be more specific, we prove that, when ε is small, just one iteration is required to achieve the desired accuracy. We then extend the method to a system of singularly perturbed parabolic problems of reaction–diffusion type. Numerical experiments support the theoretical results and demonstrate the effectiveness of the method.

1. Introduction

We consider singularly perturbed parabolic reaction–diffusion problem of the form

$$\begin{cases} \mathcal{L}u := \partial_t u - \varepsilon \partial_x^2 u + bu = f & \text{in } Q := \Omega \times (0,T] = (0,1) \times (0,T], \\ u(x,0) = 0 & \text{in } \overline{\Omega}, \\ u(0,t) = \gamma_0(t), \qquad u(1,t) = \gamma_1(t) & \text{in } (0,T], \end{cases} \tag{1}$$

where ε is arbitrary small parameter, such that $0 < \varepsilon \ll 1$, and $b \geq \alpha$ on $\overline{Q}, \alpha > 0$. We assume sufficient regularity and compatibility conditions on the data of (1) in order that $u \in C^{4,2}(\overline{Q})$. Here $C^{s,p}(D)$ denotes the space of functions with continuous spatial derivatives up to order s and continuous time derivatives up to order p on $D \subset \overline{Q}$. In particular, we assume the following compatibility conditions

$$\gamma_0^{(s)}(0) = \gamma_1^{(s)}(1) = 0 \quad \text{for } s = 0, 1, 2,$$

and

$$\partial_x^s \partial_t^q f(0,0) = \partial_x^s \partial_t^q f(1,0) = 0 \quad \text{for } 0 \leq s + 2q \leq 2.$$

Then, the solution u of (1) satisfies (see [1])

$$\|\partial_x^s \partial_t^q u\|_{\overline{Q}} \leq C\varepsilon^{-s/2} \quad \text{for } 0 \leq s + 2q \leq 4. \tag{2}$$

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Furthermore, u can be decomposed into two parts: $u = v + w$, where

$$\|\partial_x^s v\|_{\overline{Q}} \le C(1 + \varepsilon^{1-s/2}), \tag{3}$$

$$|\partial_x^s w(x,t)| \le C\varepsilon^{-s/2}\left(\exp(-x\sqrt{\alpha/\varepsilon}) + \exp(-(1-x)\sqrt{\alpha/\varepsilon})\right), \tag{4}$$

for $(x,t) \in \overline{Q}$, $s = 0, \ldots, 4$.

Classical numerical methods for solving singularly perturbed problems are known to be inadequate as they require extremely large number of mesh points to produce satisfactory numerical approximations. This leads to the development of special numerical methods – so called 'parameter-robust' or 'uniformly convergent' numerical methods – that yield approximations whose accuracy can be guaranteed independent of the perturbation parameter. A general description on parameter-robust numerical methods for such problems can be found in [2,3]. Parameter-robust numerical methods for problem (1) are discussed in [1,4–6], and the references therein. All these works involve the use of standard finite difference methods on specially designed meshes.

An overlapping Schwarz method for singularly perturbed second order ordinary differential equations is designed and analysed in [7]. Domain decomposition methods for solving partial differential equations have received much attention over the last two decades, see for example [8–11] and the references therein. The mathematical roots of the method trace back to the seminal work of Schwarz [12] in the nineteenth century. However, the general development of domain decomposition methods occurred only subsequent to the development of parallel computer architectures. More details on domain decomposition methods can be found in [11], and the references therein.

In the current work, we aim to design and analyse an overlapping Schwarz domain decomposition method for solving singularly perturbed parabolic partial differential equation (1). We use the asymptotic behaviour of the exact solution for domain partitioning. The method partitions the original domain of computation into three overlapping subdomains. On each subdomain we consider a numerical scheme combining the backward Euler scheme and central differencing. We prove that the method gives uniform numerical approximations of first order in time and almost second order in space. Furthermore, we prove that, when ε is small, just one iteration is required to achieve the desired accuracy. We then extend our method to systems of $n(\ge 2)$ singularly perturbed parabolic partial differential equations of reaction–diffusion type.

The paper is organized as follows. In Section 2 we propose an overlapping Schwarz domain decomposition method for solving problem (1). The convergence analysis is given in Section 3. In Section 4 we extend our domain decomposition method to systems of n equations. Finally, results of numerical experiments are presented in Section 5.

Notation: Throughout the paper C, sometimes subscripted, is a generic positive constant that is independent of the perturbation parameter ε, the iteration parameter k and discretization parameters N and Δt. Similarly, $\boldsymbol{C} = C(1, 1, \ldots, 1)^T$. Define $\boldsymbol{v} \le \boldsymbol{w}$ if $v_i \le w_i$, $1 \le i \le n$, and $|\boldsymbol{v}| = (|v_1|, \ldots, |v_n|)^T$. For any functions g, $y_p \in C(\overline{\Omega})$, define $g_{i,j} = g(x_i, t_j)$, $y_{p;i,j} = y_p(x_i, t_j)$; if $\boldsymbol{g}, \boldsymbol{y}_p \in C(\overline{\Omega})^n$ then $\boldsymbol{g}_{i,j} = \boldsymbol{g}(x_i, t_j) = (g_{1;i,j}, \ldots, g_{n;i,j})^T$, $\boldsymbol{y}_{p;i,j} = \boldsymbol{y}_p(x_i, t_j) = (y_{p;1;i,j}, \ldots, y_{p;n;i,j})^T$. For a vector $\boldsymbol{g} = (g_1, \ldots, g_n)^T$, define $\|\boldsymbol{g}\| = \max_{i=1,\ldots,n} |g_i|$. We consider the maximum norm and denote it by $\|\cdot\|_D$, where D is a closed and bounded set. For any mesh function Z, $\mathit{l}_j Z$ denotes the piecewise linear interpolant at time t_j.

2. Domain decomposition algorithm

We start with the decomposition of the original domain Q into three overlapping subdomains Q_p, $p = \ell, m, r$:

$$Q_\ell = \Omega_\ell \times (0,T], \qquad Q_m = \Omega_m \times (0,T], \qquad Q_r = \Omega_r \times (0,T],$$

where $\Omega_\ell = (0, 2\sigma)$, $\Omega_m = (\sigma, 1-\sigma)$ and $\Omega_\ell = (1-2\sigma, 1)$ with

$$\sigma = \min\left\{\frac{1}{4},\ 2\sqrt{\frac{\varepsilon}{\alpha}}\ \ln N\right\}. \tag{5}$$

On each subdomain $Q_p = \Omega_p \times (0,T] = (a,d) \times (0,T]$, we place a uniform mesh $\overline{\Omega}_p^N : a = x_0 < x_1 < \cdots < x_{N-1} < x_N = d$ in space and a uniform mesh $\overline{\omega}^M : 0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$ in time. Let $h_p = (d-a)/N$ be the mesh length in spatial dimension and that $\Delta t = T/M$ is the time step size. Introducing $\Omega_p^N = \overline{\Omega}_p^N \cap \Omega_p$ and $\omega^M = \overline{\omega}^M \cap (0,T]$, the mesh $Q_p^{N,M}$ on Q_p is defined by the tensor product

$$Q_\ell^{N,M} = \Omega_\ell^N \times \omega^M, \qquad Q_m^{N,M} = \Omega_m^N \times \omega^M, \qquad Q_r^{N,M} = \Omega_r^N \times \omega^M.$$

The discretization on each subdomain $Q_p^{N,M}$, $p = \ell, m, r$ is

$$[\mathcal{L}_p^{N,M} U_p]_{i,j} := [\delta_t U_p]_{i,j} - \varepsilon[\delta_x^2 U_p]_{i,j} + b_{i,j} U_{p;i,j} = f_{i,j}, \tag{6}$$

where

$$[\delta_t U_p]_{i,j} := (U_{p;i,j} - U_{p;i,j-1})/\Delta t$$