Journal of Computational and Applied Mathematics 261 (2014) 158–171

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A numerical method for constructing the Pareto front of multi-objective optimization problems

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ARTICLE INFO

Article history:
Received 8 August 2011
Received in revised form 26 January 2013

Keywords:
Multi-objective optimization
Pareto front
Pascoletti and Serafini scalarization
Nonlinear optimization
Nonconvex optimization

ABSTRACT

In this paper, a new numerical method is presented for constructing an approximation of the Pareto front of multi-objective optimization problems. This method is based on the well-known scalarization approach by Pascoletti and Serafini. The proposed method is applied to four test problems that illustrate specific difficulties encountered in multi-objective optimization problems, such as nonconvex, disjoint and local Pareto fronts. The effectiveness of the proposed method is demonstrated by comparing it with the NSGA-II algorithm and the Normal Constraint method.

1. Introduction

Multi-objective optimization is the process of simultaneously optimizing two or more objectives subject to certain constraints.

We consider a multi-objective optimization problem (MOP) as follows:

$$\textbf{\textit{MOP}} : \min_{x\in X} f(x) = (f_1(x), f_2(x), \ldots, f_p(x)), \tag{1}$$

where $X \subset \mathbb{R}^n$ is a nonempty feasible set, and f is a vector-valued function composed of p $(p \geq 2)$ real-valued functions. The image of X under f is denoted by $Y := f(X) \subseteq \mathbb{R}^p$ and referred to as the image space.

For $y, \hat{y} \in \mathbb{R}^p$,

- $y < \hat{y}$ means $y_k < \hat{y}_k$ for all $k = 1, \ldots, p$,
- $y \leqq \hat{y}$ means $y_k \leq \hat{y}_k$ for all $k = 1, \ldots, p$,
- $y \leq \hat{y}$ means $y \leqq \hat{y}$ but $y \neq \hat{y}$.

In this paper, we use the above componentwise orders to order the objective space and define the cone $\mathbb{R}^p_{\geqq} = \{x \in \mathbb{R}^p | x \geqq 0\}$.

Definition 1.1. A feasible solution $\hat{x} \in X$ is called an efficient solution of MOP (1) if there is no $x \in X$ such that $f(x) \leq f(\hat{x})$. If $\hat{x} \in X$ is efficient then $f(\hat{x})$ is called a nondominated point.

Definition 1.2. A feasible solution $\hat{x} \in X$ is called a weakly efficient solution of MOP (1) if there is no $x \in X$ such that $f(x) < f(\hat{x})$. If $\hat{x} \in X$ is weakly efficient then $f(\hat{x})$ is called a weakly nondominated point.

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0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.11.007

The set of all efficient and weakly efficient solutions of MOP (1) are denoted by X_E and X_{WE}, respectively, and their images are called Pareto front and weak Pareto front, denoted by Y_N and Y_{WN}, respectively.

Definition 1.3. The point $f^* = (f_1^*, \dots, f_p^*)^T$, where $f_i^* = \min_{x \in X} f_i(x)$ for $i = 1, \dots, p$, is called the Ideal point of MOP (1).

A common approach for determining the solutions of a MOP is reformulating it to a parameters dependent scalar optimization problem. By solving the scalar problem for a variety of parameters, several solutions of the MOP are generated. For a survey on scalarizing (and non-scalarizing) techniques, the reader is referred to [1]. There has been a great deal of effort by researchers in developing methods to generate an approximation of the Pareto front, see e.g. [2–11]. In most of these works, various types of scalarization are considered. In this paper, we concentrate on the Pascoletti and Serafini scalarization [12], and present a numerical method based on it for constructing an approximation of the Pareto front of general MOPs (nonlinear and nonconvex). An advantage of the Pascoletti and Serafini scalarization is that it is very general in the sense that many other scalarization approaches such as the weighted Tchebycheff method and the ε-constraint method can be seen as a special case of it; see [6]. Further, it is defined for general partial orderings rather than the natural ordering [12]. Our proposed method shares conceptual similarity with the approach proposed in [6], in its use of the Pascoletti and Serafini scalarization.

The Pascoletti and Serafini scalarization has two parameters named a and r, selected from $\mathbb{R}^p$. The method proposed in [6] attempts to limit the choices of a from $\mathbb{R}^p$, and still obtain all the efficient solutions. This method might be hard to verify in practice, since to implement the approach in [6], one needs to obtain a restricted hyperplane in the objective space where the parameter a changes on it. Obtaining this restricted hyperplane is simple for two objective problems, but in three and more than three objective problems is hard in practice. In comparison with the method in [6], our method can locate the Pareto front quite easily in most cases. In Section 2.2 the method in [6] is briefly discussed (for more information see [6, pp. 31–47]).

We proceed as follows. In Section 2, the Pascoletti and Serafini scalarization is briefly reviewed and some of its properties are given. In Section 3, the new parameter restriction for the Pascoletti and Serafini scalarization is proposed, and based on it the Pareto front is approximated. In Section 4, the proposed method is applied to four test problems and the results are compared with the results from NSGA-II and NC methods. Finally, conclusions are presented in Section 5.

2. Pascoletti and Serafini scalarization

In this section, a brief review of the Pascoletti and Serafini scalarization [12] is given. Pascoletti and Serafini propose the following scalar optimization problem with parameters $a, r \in \mathbb{R}^p$, in order to characterize weakly efficient and efficient solutions of MOP (1) w.r.t. the ordering cone K:

$$\begin{aligned} &\min t \\ &\text{s.t.} \\ &a + tr - f(x) \in K, \quad (SP(a, r)) \\ &x \in X, \\ &t \in \mathbb{R}. \end{aligned}$$

In this paper we assume $K = \mathbb{R}^p_{\geqq}$.

In order to solve the $SP(a, r)$ problem, the ordering cone $-\mathbb{R}^p_{\geqq}$ is moved in direction r (or $-r$) on the line $a + tr$ starting in the point a until the set $(a + tr - \mathbb{R}^p_{\geqq}) \bigcap f(X)$ is reduced to the empty set. The smallest value $\bar{t}$ for which $(a + \bar{t}r - \mathbb{R}^p_{\geqq}) \bigcap f(X) \neq \emptyset$ is the minimal value of $SP(a, r)$ [6]. This is illustrated in Fig. 1 for a two-objective problem.

2.1. Properties of the Pascoletti and Serafini scalarization

The Pascoletti and Serafini scalarization is very general, in the sense that many other scalarization approaches such as the ε-constraint method and the weighted Tchebycheff method can be seen as a special case of it; see [6]. For example, the formulation of the weighted Tchebycheff method is as follows:

$$\min_{x \in X} \max_{i=1,\dots,p} w_i(f_i(x) - a_i),$$

where $w_i > 0$ for $i = 1, \dots, p$ and $a \in \mathbb{R}^p$. If we set $K = \mathbb{R}^p_{\geqq}$ and $r_i = \frac{1}{w_i}$, then the weighted Tchebycheff method and the Pascoletti and Serafini scalarization coincide.

In the following, Section b of Theorem 2.1 in [6] is given.

Theorem 2.1. *Let $\hat{x}$ be an efficient solution of MOP (1), then $(0, \hat{x})$ is an optimal solution of $SP(a, r)$ for the parameters $a = f(\hat{x})$ and arbitrary $r \in \mathbb{R}^p_{\geqq} \setminus \{0\}$.*

Remark. Note that, as mentioned in [6], it is possible that for some parameters $a, r \in \mathbb{R}^p$ the problem $SP(a, r)$ is unbounded from below.