Journal of Computational and Applied Mathematics 261 (2014) 183–200

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Expanding the applicability of Newton's method using Smale's α-theory

Ioannis K. Argyros [a,*], Saïd Hilout [b], Sanjay K. Khattri [c]

[a] Cameron University, Department of Mathematical Sciences, Lawton, OK 73505, USA

[b] UQAM, Département des Sciences de la Terre et de l'Atmosohère, C.P. 8888, Succursale Centre-ville, Montréal (Québec) H3C 3P8, Canada

[c] Stord Haugesund University College, Haugesund, Norway

ARTICLE INFO

Article history:
Received 28 March 2012
Received in revised form 29 October 2013

Keywords:
Newton's method
Banach space
Semi-local convergence
Smale's α-theory
Fréchet-derivative

ABSTRACT

We present a tighter convergence analysis than earlier studies such as in Cianciaruso (2007), Guo (2007), Shen and Li (2010), Smale (1986, 1987), Wang and Zhao (1995), Wang (1999), Wang and Han (1990) of Newton's method using Smale's α-theory by introducing the notion of the center γ_0-condition. In particular, in the semilocal convergence case we show that if the center γ_0-condition is smaller than the γ-condition, then the new majorizing sequence is tighter than the old majorizing sequence. The new convergence criteria are weaker than the older convergence criteria. Furthermore, in the local convergence case, we obtain a larger radius of convergence and tighter error estimates on the distances involved. These improvements are obtained under the same computational cost. Numerical examples and applications are also provided in this study to show that the older results cannot apply but the new results apply to solve equations.

1. Introduction

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. Let $U(x, r)$ and $\overline{U}(x, r)$ stand, respectively, for the open and the closed ball in $\mathcal{X}$ with center x and radius $r > 0$. Denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. In the present paper we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1}$$

where F is a Fréchet continuously differentiable operator defined on $\overline{U}(x_0, R)$ for some $R > 0$ with values in $\mathcal{Y}$.

A lot of problems from computational sciences and other disciplines can be brought in the form of Eq. (1) using Mathematical Modelling [1,2]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [1–4]. The study about convergence matter of Newton methods is usually centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of Newton methods; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. We find in the literature several studies on the weakness and/or extension of the hypothesis made on the underlying operators. There is a plethora on local as well as semi-local convergence results; we refer the reader to [5–8,1,9–15,2,16–22,3,4,23–34]. The most famous among the semi-local convergence of iterative methods is the celebrated Kantorovich theorem for solving

* Corresponding author. Tel.: +1 580 581 2908; fax: +1 580 5812616.
E-mail addresses: iargyros@cameron.edu (I.K. Argyros), said.hilout@math.univ-poitiers.fr (S. Hilout), sanjay.khattri@hsh.no (S.K. Khattri).

0377-0427/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cam.2013.10.054

nonlinear equations. This theorem provides a simple and transparent convergence criterion for operators with bounded second derivatives F'' or the Lipschitz continuous first derivatives [1–4]. Another important theorem inaugurated by Smale at the International Conference of Mathematics (cf. [28]), where the concept of an approximate zero was proposed and the convergence criteria were provided to determine an approximate zero for the analytic function, depending on the information at the initial point. Wang and Han [32,31] generalized Smale's result by introducing the γ-condition (see (3)). For more details on Smale's theory, the reader can refer to the excellent Dedieu's book [17, Chapter 3.3].

Newton's method defined by

$$\begin{aligned} &x_0 \text{ is an initial point} \\ &x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad \text{for each } n = 0, 1, 2, \ldots \end{aligned} \tag{2}$$

is undoubtedly the most popular iterative process for generating a sequence $\{x_n\}$ approximating $x^\star$. Here, $F'(x)$ denotes the Fréchet-derivative of F at $x \in \overline{U}(x_0, R)$.

In the present paper we expand the applicability of Newton's method under the γ-condition by introducing the notion of the center γ_0-condition (to be precised in Definition 3.1) for some $\gamma_0 \le \gamma$. This way we obtain tighter upper bounds on the norms of $\|F'(x)^{-1} F'(x_0)\|$ for each $x \in \overline{U}(x_0, R)$ (see (15), (13) and (14)) leading to weaker sufficient convergence conditions and a tighter convergence analysis than in earlier studies such as [16,21,27,28,31,32]. The approach of introducing the center-Lipschitz condition has already been fruitful for expanding the applicability of Newton's method under the Kantorovich-type theory [7,12,14,2].

Wang in his work [31] on approximate zeros of Smale (cf. [28,29]) used the γ-Lipschitz condition at x_0

$$\|F'(x_0)^{-1} F''(x)\| \le \frac{2\gamma}{(1 - \gamma\, \|x - x_0\|)^3} \quad \text{for each } x \in U(x_0, r),\ 0 < r \le R, \tag{3}$$

where $\gamma > 0$ and x_0 are such that $\gamma\, \|x - x_0\| < 1$ and $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ to show the following semi-local convergence result for Newton's method.

Theorem 1.1. *Let $F : \overline{U}(x_0, R) \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be twice-Fréchet differentiable. Suppose there exists $x_0 \in U(x_0, R)$ such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and*

$$\|F'(x_0)^{-1} F(x_0)\| \le \eta; \tag{4}$$

condition (3) holds and for $\alpha = \gamma\, \eta$

$$\alpha \le 3 - 2\sqrt{2}; \tag{5}$$

$$t^\star \le R, \tag{6}$$

where

$$t^\star = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma} \le \left(1 - \frac{1}{\sqrt{2}}\right) \frac{1}{\gamma}. \tag{7}$$

Then, sequence $\{x_n\}$ generated by Newton's method is well defined, remains in $\overline{U}(x_0, t^\star)$ for each $n = 0, 1, \ldots$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $F(x) = 0$. Moreover, the following error estimates hold:

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \tag{8}$$

and

$$\|x_{n+1} - x^\star\| \le t^\star - t_n, \tag{9}$$

where scalar sequence $\{t_n\}$ is defined by

$$\begin{aligned} &t_0 = 0, \quad t_1 = \eta, \\ &t_{n+1} = t_n + \frac{\gamma\, (t_n - t_{n-1})^2}{\left(2 - \frac{1}{(1-\gamma\, t_n)^2}\right)(1 - \gamma t_n)(1 - \gamma t_{n-1})^2} = t_n - \frac{\varphi(t_n)}{\varphi'(t_n)} \quad \text{for each } n = 1, 2, \ldots, \end{aligned} \tag{10}$$

where

$$\varphi(t) = \eta - t + \frac{\gamma\, t^2}{1 - \gamma\, t}. \tag{11}$$

Notice that $t^\star$ is the small zero of equation $\varphi(t) = 0$, which exists under the hypothesis (5).

The paper is organized as follows: Sections 2 and 3 contain the semi-local and the local convergence analysis of Newton's method. Applications and numerical examples are given in Section 4.