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A dimension by dimension splitting immersed interface method for heat conduction equation with interfaces ☆

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ABSTRACT

The numerical method proposed in this paper is an improvement of the ADI method by Li and Mayo (1994). The proposed method is unconditionally stable for both two and three-dimensional heat conduction interface problems, while Li's ADI method is only stable for two-dimensional problems. The method is a modification of a Locally One-Dimensional (LOD) difference scheme, with correction term added to the right-hand side of the standard LOD difference scheme at irregular points. The correction term is determined so that the local truncation error is of order $O(h)$ at irregular points. Then the method is two-order convergent in both time and space directions. Numerical examples show good agreement with exact solutions and confirm the order of convergence and stability.

1. Introduction

Interface problems are those problems in which discontinuous coefficients or singular sources are presented across some interfaces. The solutions of these problems are often non-smooth or even discontinuous across the interface. Hence, for interface problems, the accuracy of traditional difference scheme may be degraded because the Taylor expansions that they are based on are not valid across the interface.

In 1994, Leveque and Li [1,2] proposed an efficient numerical method, the Immersed Interface Method (IIM), for elliptic interface problems. It is a method with two-order accuracy in maximum norm. The difference scheme of IIM is the same as the standard difference scheme away from the interface. When the interface passes through the difference stencil, the difference scheme should be modified by incorporating the jump condition of variables across the interface. However, in two-dimensional problems another grid point should be added to the standard five-point difference stencil, leading to a non-symmetric coefficient matrix. This problem was improved by introducing the maximum principle preserving immersed interface method [3]. It produces a diagonally dominant coefficient matrix and has been successfully implemented with a specially designed multigrid method [4]. The explicit jump immersed interface method (EJIIM) was proposed in [5] for two-dimensional elliptic interface problems, which produces a symmetric linear system by adding proper correction terms on the right-hand side (RHS) of the difference equation. The decomposed IIM [6] was proposed to handle variable coefficient elliptic interface problems. It decomposes the jump condition into each direction and produces a symmetric problem. Li proposed a fast iterative immersed interface method (FIIIM) in [7] for solving elliptic interface problems with discontinuous

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coefficients. The convergence analysis [8,9] shows that the local truncation error is allowed to be of lower order along some lower dimensional manifolds without influencing the global convergence order of the solution.

For parabolic interface problems, IIM has been extended to deal with linear parabolic interface problems [4], in which the Crank–Nicolson scheme is used to treat the diffusion term. Li considered the nonlinear parabolic interface problems in [10], mainly focusing on how to treat the moving interface. After that, Wiegmann and Bube [11] investigated the same nonlinear parabolic interface problems, handling the nonlinear convection term in a convenient way under the IIM strategy. The immersed interface based θ-difference scheme was proposed for two dimensional heat-diffusion equations with singular own sources [12]. A stable and fast prediction–correction implicit scheme [13] was developed to handle the thermo-elastic system with discontinuities.

For time-dependent problems, implicit schemes are popular because they can eliminate the time step restriction. One disadvantage is that a large linear system of equations needs solving at each time step. The approach to overcome this drawback may be to use the Alternative Direction Implicit (ADI) or splitting method, in which only a sequence of tri-diagonal systems needs solving. The P-R ADI scheme [14] and the LOD scheme [15,16] are of these methods. In [17,18], the P-R ADI scheme was applied to heat equations and Stefan problems, respectively. However, Li's modified P-R ADI scheme is only absolutely stable for two-dimensional heat conduction problems, while the LOD scheme can be extended directly from 2D to 3D without losing the merit of absolute stability [19,20]. The intention of this paper is to apply LOD schemes to heat interface problems in 2D and 3D. The main objective is to add a correction term to the RHS of the standard difference scheme at irregular points, thus it will not affect the stability of standard LOD schemes. The level set function [21] is used to represent the interface due to its simplicity and strength in describing complex shapes.

The rest of this paper is organized as follows: In Section 2 we describe the problems to be solved and provide some preparations for this paper. Then, the construction and analysis of LOD–IIM in 2D are presented in Section 3. A similar extension of LOD–IIM to 3D problem is shown in Section 4. We display some numerical examples in Section 5 before a concluding remark is made in Section 6.

2. Preliminaries

2.1. Problem description

We consider the following heat conduction equation

$$\frac{\partial u}{\partial t} = \Delta u - f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, T], \tag{2.1}$$

with initial condition,

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.2}$$

and boundary condition,

$$u(\mathbf{x}, t) = g(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T], \tag{2.3}$$

where Δ is the Laplace operator in 2D or 3D, with $\mathbf{x} = (x, y)$ in 2D and $\mathbf{x} = (x, y, z)$ in 3D. The source $f(\mathbf{x}, t)$ and initial value $u_0(\mathbf{x})$ may be discontinuous or singular across interface Γ. The interface Γ is immersed in the domain Ω and divides Ω into two parts, Ω^+ and Ω^-. Due to the bad initial value and source term, the solution $u(\mathbf{x}, t)$ will be discontinuous or non-smooth across the interface. Along the interface, the jump conditions $[u]$ and $[u_\mathbf{n}]$ can often be derived from the problem itself,

$$[u] = \lim_{\substack{\mathbf{x}\to\Gamma \\ \mathbf{x}\in\Omega^+}} u(\mathbf{x}, t) - \lim_{\substack{\mathbf{x}\to\Gamma \\ \mathbf{x}\in\Omega^-}} u(\mathbf{x}, t) = w(\mathbf{x}, t), \tag{2.4}$$

$$[u_\mathbf{n}] = \lim_{\substack{\mathbf{x}\to\Gamma \\ \mathbf{x}\in\Omega^+}} u_\mathbf{n}(\mathbf{x}, t) - \lim_{\substack{\mathbf{x}\to\Gamma \\ \mathbf{x}\in\Omega^-}} u_\mathbf{n}(\mathbf{x}, t) = v(\mathbf{x}, t), \tag{2.5}$$

where $\mathbf{n}$ is the unit outward normal vector, see Fig. 1.

2.2. Level set function

Since the interface can have a fairly complex shape, a smooth level set function $\phi(\mathbf{x})$ is introduced as

$$\phi(\mathbf{x}) = \pm dis,$$

where dis is the shortest distance from $\mathbf{x}$ to the interface Γ. Thus the set of points with $\phi(\mathbf{x}) = 0$ represents the interface Γ,

$$\Gamma = \{\mathbf{x} \in \mathbb{R}^{dim} \mid \phi(\mathbf{x}) = 0\}, \quad dim = 2, 3. \tag{2.6}$$

The set of all points with $\phi(\mathbf{x}) < 0$ and the set of all points with $\phi(\mathbf{x}) > 0$ represent two disjoint domains Ω^- and Ω^+, respectively. The unit outward normal vector $\mathbf{n}$ can be approximated by $\mathbf{n} = \nabla\phi/|\nabla\phi|$ at any point.