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A new exact penalty method for semi-infinite programming problems

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ABSTRACT

In this paper, we consider a class of nonlinear semi-infinite optimization problems. These problems involve continuous inequality constraints that need to be satisfied at every point in an infinite index set, as well as conventional equality and inequality constraints. By introducing a novel penalty function to penalize constraint violations, we form an approximate optimization problem in which the penalty function is minimized subject to only bound constraints. We then show that this penalty function is exact—that is, when the penalty parameter is sufficiently large, any local solution of the approximate problem can be used to generate a corresponding local solution of the original problem. On this basis, the original problem can be solved as a sequence of approximate nonlinear programming problems. We conclude the paper with some numerical results demonstrating the applicability of our approach to PID control and filter design.

1. Introduction

In this paper, we consider semi-infinite programming problems of the following form:

$$\text{minimize } f(\boldsymbol{x}) \tag{1a}$$

$$\text{subject to } \varphi_i(\boldsymbol{x}, \omega) \le 0, \quad \omega \in \Omega_i,\ i \in \mathcal{C}, \tag{1b}$$

$$g_i(\boldsymbol{x}) \le 0, \quad i \in \mathcal{I}, \tag{1c}$$

$$h_i(\boldsymbol{x}) = 0, \quad i \in \mathcal{E}, \tag{1d}$$

$$a_j \le x_j \le b_j, \quad j = 1, \ldots, n, \tag{1e}$$

where $\boldsymbol{x} = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ is the decision vector; $f, g_i, h_i : \mathbb{R}^n \to \mathbb{R}$ and $\varphi_i : \mathbb{R}^n \times \Omega_i \to \mathbb{R}$ are continuously differentiable functions; a_j and b_j are given constants satisfying $a_j < b_j$; and $\Omega_i \subset \mathbb{R}$ are compact intervals of positive measure. We refer to this problem as Problem (P).

If $\mathcal{C} = \emptyset$, then Problem (P) is a standard nonlinear programming problem that can be solved efficiently using well-known methods such as sequential quadratic programming (see [1,2]). Thus, the main difficulty with Problem (P) is the continuous inequality constraints (1b), which arise in a wide range of important applications such as signal processing [3], circuit design [4,5], and optimal control [6,7]. Each continuous inequality constraint in (1b) actually defines an infinite number of constraints—one for each point in Ω_i.

Teo and Goh in [8] have proposed a simple approach for tackling Problem (P). This approach involves transforming the continuous inequality constraints (1b) into the following set of equivalent equality constraints:

$$c_i \int_{\Omega_i} \left[\max\{\varphi_i(\boldsymbol{x}, \omega), 0\}\right]^2 d\omega = 0, \quad i \in \mathcal{C}, \tag{2}$$

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where $c_i > 0$, $i \in \mathcal{C}$ are given weights. Thus, the continuous inequality constraints are replaced by a finite set of equality constraints, and the resulting optimization problem can, in principle, be solved using conventional techniques. The downside of this approach, however, is that the equality constraints (2) do not satisfy the standard linear independence constraint qualification, and thus numerical convergence cannot be guaranteed.

To overcome this limitation, Jennings and Teo in [9] proposed an alternative method in which the continuous inequality constraints (1b) are approximated as follows:

$$\int_{\Omega_i} \mathcal{L}_\epsilon(\varphi_i(\boldsymbol{x}, \omega))d\omega \leq \tau, \quad i \in \mathcal{C}, \tag{3}$$

where $\epsilon > 0$ and $\tau > 0$ are adjustable parameters and $\mathcal{L}_\epsilon : \mathbb{R} \to \mathbb{R}$ is a smooth approximation of $\max\{\cdot, 0\}$. The approximating function $\mathcal{L}_\epsilon$ is specially designed so that $\mathcal{L}_\epsilon(\eta) \geq 0$ and $\mathcal{L}_\epsilon(\eta) \to \max\{\eta, 0\}$ as $\epsilon \to 0$. Replacing (1b) with (3) yields an approximate nonlinear programming problem. It can be shown that for each $\epsilon > 0$, if τ is chosen sufficiently small, then any solution of the approximate problem is feasible for Problem (P). Furthermore, the optimal cost of the approximate problem converges to the optimal cost of Problem (P) as $\epsilon \to 0$. On this basis, a solution of Problem (P) can be obtained by solving a sequence of approximate problems, where the parameters ϵ and τ are adjusted appropriately according to certain rules.

This idea was further developed in [10] with the introduction of the following penalty function, which is based on the constraint approximation (3):

$$f(\boldsymbol{x}) + \sigma \sum_{i \in \mathcal{C}} \int_{\Omega_i} \mathcal{L}_\epsilon(\varphi_i(\boldsymbol{x}, \omega))d\omega, \tag{4}$$

where $\sigma > 0$ is the penalty parameter. Note that violations of the continuous inequality constraints (1b) are penalized by the integral term in (4). It can be shown that for each $\epsilon > 0$, if σ is made sufficiently large, then any minimizer of (4) on the region defined by (1c)–(1e) is feasible for Problem (P). Thus, a solution of Problem (P) can be obtained by minimizing (4) for appropriate choices of the parameters ϵ and σ.

Although the constraint approximation methods in [9,10] generally perform well, numerical convergence is only guaranteed when the approximate problems are solved in a global sense. However, in practice, the approximate problems (and the original problem) are usually non-convex, and thus we can only expect to solve them locally. Unfortunately, conditions under which a local solution of the approximate problem converges to a local solution of the original problem are not known.

Motivated by this drawback, Yu et al. in [11,12] recently introduced a new penalty function defined as follows:

$$F(\boldsymbol{x}, \epsilon) \triangleq \begin{cases} f(\boldsymbol{x}), & \text{if } \epsilon = 0, \quad \Delta(\boldsymbol{x}, \epsilon) = 0, \\ f(\boldsymbol{x}) + \epsilon^{-\alpha}\Delta(\boldsymbol{x}, \epsilon) + \sigma\epsilon^{\beta}, & \text{if } \epsilon > 0, \\ \infty, & \text{otherwise,} \end{cases} \tag{5}$$

where

$$\Delta(\boldsymbol{x}, \epsilon) \triangleq \sum_{i \in \mathcal{C}} \int_{\Omega_i} \left[\max\{\varphi_i(\boldsymbol{x}, \omega) - \epsilon^{\gamma} W_i, 0\}\right]^2 d\omega. \tag{6}$$

Here, ϵ is a new decision variable, $\sigma > 0$ is the penalty parameter, and $\alpha > 0$, $\beta > 2$, $\gamma > 0$, and $W_i \in (0, 1)$, $i \in \mathcal{C}$ are fixed constants.

Unlike (4), the penalty function (5) only involves one adjustable parameter (the penalty parameter σ). Furthermore, when σ is sufficiently large (and certain technical conditions are satisfied), any local minimizer of (5) can be used to generate a corresponding local minimizer of Problem (P) (with $\mathcal{E} = \mathcal{I} = \emptyset$). This result is more practical than the convergence results given in [9,10], which are only applicable when the approximate problems are solved globally.

The penalty function (5) is a clear improvement over (4). However, it still has two disadvantages:

(i) Eqs. (5) and (6) involve $|\mathcal{C}| + 3$ fixed parameters, each of which needs to be selected judiciously.
(ii) Convergence is only guaranteed when there are no standard equality or inequality constraints (i.e. $\mathcal{I} = \mathcal{E} = \emptyset$) and none of the bound constraints are active at an optimal solution.

The aim of this paper is to address these issues by proving new convergence results under less stringent conditions. In particular, we will show that the parameters W_i in (6) are actually unnecessary, and (5) is still an effective penalty function when $W_i = 0$. Accordingly, the number of fixed parameters in the penalty function can be significantly reduced from $|\mathcal{C}|+3$ to just 2. This simplified penalty function is still exact in the sense that the penalty parameter is not required to reach infinity for the constraints in Problem (P) to be satisfied. Furthermore, the numerical results in Section 5 show that our simplified penalty function is just as effective as the original one proposed in [11,12].