Journal of Computational and Applied Mathematics 261 (2014) 352–363

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Generalized Bézier curves and surfaces based on Lupaş q-analogue of Bernstein operator

Li-Wen Han [a,b,*], Ying Chu [a], Zhi-Yu Qiu [a]

[a] College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei 050024, PR China

[b] Hebei Province Key Laboratory of Computational Mathematics and Application, Shijiazhuang, Hebei 050024, PR China

ARTICLE INFO

Article history:
Received 12 April 2013
Received in revised form 14 November 2013

Keywords:
Lupaş q-analogue of Bernstein operator
Lupaş q-Bézier curve
Lupaş q-Bézier surface
Degree elevation
de Casteljau algorithm
Shape parameter

ABSTRACT

In this paper, a new generalization of Bézier curves with one shape parameter is constructed. It is based on the Lupaş q-analogue of Bernstein operator, which is the first generalized Bernstein operator based on the q-calculus. The new curves have some properties similar to classical Bézier curves. Moreover, we establish degree evaluation and de Casteljau algorithms for the generalization. Furthermore, we construct the corresponding tensor product surfaces over the rectangular domain, and study the properties of the surfaces, as well as the degree evaluation and de Casteljau algorithms. Compared with q-Bézier curves and surfaces based on Phillips q-Bernstein polynomials, our generalizations show more flexibility in choosing the value of q and superiority in shape control of curves and surfaces. The shape parameters provide more convenience for the curve and surface modeling.

1. Introduction

A century ago, S.N. Bernstein [1] defined his famous polynomials named Bernstein polynomials to prove the Weierstrass theorem. Due to the fine properties of approximation, convergence and shape preserving, Bernstein polynomials play important roles in approximation theory as well as analysis, geometry and computer science. The classical Bézier curve [2] constructed with Bernstein basis functions is the most important curve in computer aided geometry design (CAGD). In recent years, generalization of the Bézier curve with shape parameters has received continuous attention. Goldman and Barry [3] constructed a kind of generalized Bézier curves with one shape parameter named h-Bézier curves by using Pólya polynomials. Han et al. [4] constructed novel generalized Bézier curves by introducing some parameters to Bernstein polynomials, which they called Quasi-Bézier curves. In 2011, Chen and Wang [5] made some improvements on the Quasi-Bézier curves, so as to make them be applied more widely in CAGD. Other forms of generalized Bézier curves can be seen in [6,7]. All of them were concerned with the problem of changing the shape of curves and surfaces, while keeping the control polygon unchanged.

Recently, the rapid development of q-calculus has led to the discovery of new generalizations of Bernstein polynomials involving q-integers [8]. In 1987, Lupaş [9] introduced the first generalized Bernstein operator based on q-integers. In 1996, Phillips [10] proposed another q-variant of the classical Bernstein operator, the so-called Phillips q-Bernstein operator, which may be expressed in terms of q-difference [11] and attracted lots of investigations. The q-variants of Bernstein polynomials provide one shape parameter for constructing free-form curves and surfaces, Phillips q-Bernstein operator was applied well in this area. In 2003, Oruç and Phillips [12] used the basis functions of Phillips q-Bernstein operator for construction of q-Bézier curves, which we call Phillips q-Bézier curves here, and studied the properties of degree reduction and evaluation, as

* Corresponding author at: College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei 050024, PR China. Tel.: +86 13930143339.

E-mail addresses: hanliwen@sina.com (L.-W. Han), chuyingyouxiang@163.com (Y. Chu).

0377-0427/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cam.2013.11.016

well as the variation diminishing property. In 2007, Dişibüyük and Oruç [13] defined a q-generalization of rational Bernstein Bézier curves. In 2008, Dişibüyük and Oruç [14] defined the tensor product q-Bernstein–Bézier surfaces on $[0, 1] \times [0, 1]$ based on the Phillips q-Bernstein operator. More intensively, Simeonov et al. [15] developed blossoming and subdivision procedures for Phillips q-Bézier curves. In 2012, Simeonov and Goldman [16] defined B-splines based on q-calculus using the q-blossoming. These are called quantum B-splines.

Compared with the q-Bernstein operator, the Lupaş q-analogue of Bernstein operator has had little application in CAGD. In 2010, Phillips [17] indicated that there was no on the practical application of Lupaş q-analogue of Bernstein operator. While in approximation theory, there have been some researches of convergence and approximation on the Lupaş q-analogue of the Bernstein operator. In 1987, Lupaş [9] studied its approximating and shape preserving properties. In 2006, Ostrovska [18] investigated its uniform convergence. In 2010, Mahmudov and Sabancigil [19] further discussed its approximating problems. In 2011, Zoltan [20] showed the quantitative estimates for the Lupaş q-analogue of the Bernstein operator. All of the results on the Lupaş q-analogue of the Bernstein operator allow us to construct a new generalization of Bézier curves.

First we require some preliminaries [21]. Given a real number $q > 0$, and any $r \in N$, we define $[r]$ as

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1, \end{cases}$$

and call $[r]$ a *q-integer*.

Let $N_q = \{[r], r \in N\}$, then from the definition above, we can see that $N_q = \{0, 1, 1+q, 1+q+q^2, 1+q+q^2+q^3, \ldots\}$. Obviously, the set of q-integers N_q generalizes the set of nonnegative integers N, which we recover by putting $q = 1$.

We also define *q-factorial* [r]!, given any $r \in N$,

$$[r]! = \begin{cases} [r][r-1]\cdots[1], & r \geq 1, \\ 1, & r = 0. \end{cases}$$

The *q-binomial coefficient* $\begin{bmatrix} n \\ r \end{bmatrix}$, is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n][n-1]\cdots[n-r-1]}{[r]!} = \frac{[n]!}{[r]![n-r]!}$$

for $n \geq r \geq 1$, and has the value 1 when $r = 0$ and value 0 otherwise.

In particular, the q-binomial coefficients satisfy the Pascal-type relations:

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}, \tag{1}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ r \end{bmatrix}. \tag{2}$$

Let $f \in C[0, 1]$. The linear operator $L_{n,q} : C[0, 1] \to C[0, 1]$ is defined by

$$L_{n,q}(f; x) = \sum_{r=0}^{n} b_{n,r}(x; q) f_r$$

where

$$f_r = f\left(\frac{[r]}{[n]}\right), \qquad b_{n,r}(x; q) = \frac{a_{n,r}(x; q)}{w_n(x; q)},$$

and

$$a_{n,r}(x; q) = \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-1)/2} x^r (1-x)^{n-r},$$

$$w_n(x; q) = \sum_{r=0}^{n} a_{n,r}(x; q) = \prod_{r=1}^{n} (1 - x + q^{r-1} x).$$

$L_{n,q}$ is called the *Lupaş q-analogue of the Bernstein operator* [9].

This paper is organized as follows. In Section 2, we derive a class of functions over [0, 1] from the Lupaş q-analogue of Bernstein operator. The properties of this class of functions are shown in this section. In Section 3, the Lupaş q-Bézier curves are constructed and its properties are studied. We derive the corresponding degree evaluation and the de Casteljau algorithm. In Section 4, we define the generalized tensor product Bézier surfaces over the rectangular domain with two parameters based on the Lupaş q-analogue of the Bernstein functions, and study their properties. The effects on the shape of the curves and surfaces by the shape parameters are shown in Section 5.