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Remarks on two integral operators and numerical methods for CSIE[☆]

Maria Carmela De Bonis [*]

Department of Mathematics, Computer Science and Economics, University of Basilicata, Via dell'Ateneo Lucano n. 10, 85100 Potenza, Italy

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ABSTRACT

In this paper the author extends the mapping properties of some singular integral operators in Zygmund spaces equipped with uniform norm. As a by-product quadrature methods for solving CSIE having indices 0 and 1 are proposed. Their stability and convergence are proved and error estimates in Zygmund norm are given. Some numerical tests are also shown.

1. Introduction and preliminaries

It is well-known (see, for instance, [1]) that the operator

$$(D^{\alpha,-\alpha}f)(x) = (\cos\pi\alpha)v^{\alpha,-\alpha}(x)f(x) - \frac{\sin\pi\alpha}{\pi}\int_{-1}^{1} f(y)\frac{v^{\alpha,-\alpha}(y)}{y-x}dy,$$

with $v^{\rho,\sigma}(x) = (1-x)^\rho(1+x)^\sigma$, $\rho,\sigma > -1$, a Jacobi weight, and its inverse $D^{-\alpha,\alpha}$ are isometric maps in the couples of spaces $(L^2_w, L^2_{1/w})$ and $(L^2_{1/w}, L^2_w)$, $w = v^{\frac{\alpha}{2},-\frac{\alpha}{2}}$, respectively. Analogous properties, but under the assumption $\int_{-1}^{1} f(x)v^{-\alpha,\alpha-1}(x)dx = 0$, hold true for the operator

$$(D^{-\alpha,\alpha-1}f)(x) = (\cos\pi\alpha)v^{-\alpha,\alpha-1}(x)f(x) + \frac{\sin\pi\alpha}{\pi}\int_{-1}^{1} f(y)\frac{v^{-\alpha,\alpha-1}(y)}{y-x}dy,$$

and its inverse $D^{\alpha,1-\alpha}$ in the couples of spaces $(L^2_u, L^2_{1/u})$ and $(L^2_{1/u}, L^2_u)$, $u = v^{-\frac{\alpha}{2},\frac{\alpha-1}{2}}$, respectively.

By contrast, in the space of continuous functions in $[-1,1]$, $C^0 := C^0([-1,1])$, equipped with the uniform norm the previous operators are unbounded.

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* Tel.: +39 0971205859.
E-mail address: mariacarmela.debonis@unibas.it.

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Now, we introduce the space

$$C_{v^{\rho,\sigma}} = \left\{ f \in C^0((-1,1)) : \lim_{|x|\to 1} (f v^{\rho,\sigma})(x) = 0 \right\}, \quad \rho, \sigma \geq 0,$$

equipped with the norm

$$\|f\|_{C_{v^{\rho,\sigma}}} := \max_{|x|\leq 1} |(f v^{\rho,\sigma})(x)| = \|f v^{\rho,\sigma}\|_\infty,$$

with the obvious modifications if ρ e/o σ are zeros (letting $C_{v^{0,0}} = C^0$).

A subspace of $C_{v^{\rho,\sigma}}$ is the Sobolev space

$$W_r(v^{\rho,\sigma}) := \{f \in C_{v^{\rho,\sigma}} : f^{(r-1)} \in AC(-1,1) \text{ and } \|f^{(r)} \varphi^r v^{\rho,\sigma}\|_\infty < \infty\},$$

where $AC(-1, 1)$ is the set of all functions that are absolutely continuous in every compact set of $(-1, 1)$, equipped with the norm

$$\|f\|_{W_r(v^{\rho,\sigma})} = \|f v^{\rho,\sigma}\|_\infty + \|f^{(r)} \varphi^r v^{\rho,\sigma}\|_\infty.$$

Another subspace is the Zygmund space

$$Z_r(v^{\rho,\sigma}) := Z_{r,k}(v^{\rho,\sigma}) = \left\{ f \in C_{v^{\rho,\sigma}} : \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_{v^{\rho,\sigma}}}{t^r} < +\infty \right\}$$

equipped with the norm

$$\|f\|_{Z_r(v^{\rho,\sigma})} = \|f v^{\rho,\sigma}\|_\infty + \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_{v^{\rho,\sigma}}}{t^r},$$

where $r > 0$ is an arbitrary real number, $k > r$ is integer,

$$\Omega_\varphi^k(f,t)_{v^{\rho,\sigma}} := \sup_{0<h\leq t} \|(\Delta_{h\varphi}^k f) v^{\rho,\sigma}\|_{C(I_{h,k})}, \tag{1}$$

$$\Delta_{h\varphi}^k f(x) := \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\left(x + \frac{kh}{2}\varphi(x) - ih\varphi(x)\right),$$

$I_{h,k} := [-1 + 4k^2h^2, 1 - 4k^2h^2]$, $0 < t < 1$ and $\varphi(x) = \sqrt{1-x^2}$. We remark that $W_r(v^{\rho,\sigma}) \subseteq Z_r(v^{\rho,\sigma})$ when $r = k$.

With the above notations, in [2,3] the authors showed that $D^{\alpha,-\alpha}$ is bounded and invertible in the couple $(Z_r(v^{\alpha,0}), Z_r(v^{0,\alpha}))$ and, moreover, $D^{-\alpha,\alpha-1}$, under the assumption $\int_{-1}^{1} f(x) v^{-\alpha,\alpha-1}(x)dx = 0$, is bounded and invertible in the couple $(Z_r(v^{0,0}), Z_r(v^{\alpha,1-\alpha}))$.

As a first contribution of this paper, we prove that $D^{\alpha,-\alpha}$ is bounded and invertible in the couple $(Z_r(v^{\alpha+\gamma,\delta}), Z_r(v^{\gamma,\alpha+\delta}))$ and, analogously, $D^{-\alpha,\alpha-1}$, under the assumption $\int_{-1}^{1} f(x) v^{-\alpha,\alpha-1}(x)dx = 0$, is bounded and invertible in the couple $(Z_r(v^{\gamma,\delta}), Z_r(v^{\alpha+\gamma,1-\alpha+\delta}))$. Obviously, the parameters α, γ, δ have to satisfy suitable conditions that we will assign explicitly.

Then, the results in [2,3] have been extended to larger functional spaces. This fact suggested us to propose two numerical methods to approximate the solutions of the equations

$$(D^{\alpha,-\alpha} + K^{\alpha,-\alpha}) f(x) = g(x) \tag{2}$$

and

$$(D^{-\alpha,\alpha-1} + K^{-\alpha,\alpha-1}) f(x) = g(x), \qquad \int_{-1}^{1} f(x) v^{-\alpha,\alpha-1}(x)dx = 0, \tag{3}$$

where $K^{\alpha,-\alpha}$ and $K^{\alpha,1-\alpha}$ are compact perturbations. Eqs. (2) and (3) are well-known Cauchy equations of indices $\chi = 0$ and $\chi = 1$, respectively.

The proposed numerical methods are stable and convergent for any choice of the parameter $0 < \alpha < 1$. We would like to emphasize that the range of α is $(0, 1)$, taking into account that the numerical methods proposed in [4], using the properties proved in [2,3], lead to strong restrictions: $\frac{1}{2} \leq \alpha < 1$ for Eq. (2) and $\alpha = \frac{1}{2}$ for Eq. (3).

The paper is organized as follows. In Section 2 we give the main results. Section 3 contains the description of the quadrature methods we propose for Eqs. (2) and (3). We show that both of them are stable and convergent and lead to solve well-conditioned linear systems. The proofs of the main results are given in Section 4, while Appendix contains some proofs that are more technical. Finally, in Section 5, we show the efficiency of our procedures by some numerical tests.