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Some identities involving exponential functions and Stirling numbers and applications

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ABSTRACT

Guo and Qi (2013) posed a problem asking to determine the coefficients $a_{k,i-1}$ for $1 \le i \le k$ such that $1/(1-e^{-t})^k = 1+\sum_{i=1}^{k} a_{k,i-1}(1/(e^t-1))^{(i-1)}$. The authors answer this question alternatively by Faà di Bruno's formula, unify the eight identities due to Guo and Qi to two identities involving two parameters, and apply them to obtain an explicit expression for the Apostol–Bernoulli numbers and the Fubini numbers, respectively.

1. Introduction

Motivated by two identities in [1] the following problem was posed in the recently published paper [2] and its preprint [3]. For $t \neq 0$ and $k \in \mathbb{N}$, determine the numbers $a_{k,i-1}$ for $1 \le i \le k$ such that

$$\frac{1}{(1-e^{-t})^k} = 1 + \sum_{i=1}^{k} a_{k,i-1} \left(\frac{1}{e^t - 1} \right)^{(i-1)}. \tag{1.1}$$

Stimulated by this problem, eight identities involving the exponential function were established. Theorems 2.1 and 2.2 together with Corollaries 2.1 and 2.2 in [2] state that the derivatives $(1/(e^t-1))^{(i)}$ and $(1/(1-e^{-t}))^{(i)}$ can be expressed by linear combinations of the functions $1/(e^t-1)^k$ and $1/(1-e^{-t})^k$. They do conversely. They were proved by induction. Now we state them below. Because those corollaries presented in [2] are equivalent to the corresponding theorems, we will not repeat them here.

Theorem 1.1 *([2,3]). For $i \in \{0\} \cup \mathbb{N}$, we have*

$$\left(\frac{1}{e^t - 1} \right)^{(i)} = \sum_{k=1}^{i+1} \frac{\lambda_{i,k}}{(e^t-1)^k}, \tag{1.2}$$

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where

$$\lambda_{i,k} = (-1)^i (k-1)! S(i+1,k)$$

and

$$S(i,k) = \frac{1}{k!}\sum_{l=1}^{k}(-1)^{k-l}\binom{k}{l} l^i$$

are the Stirling numbers of the second kind.

Obviously, (1.2) can be rewritten as

$$\left(\frac{1}{1-e^t}\right)^{(i)} = \sum_{k=1}^{i+1}\frac{\nu_{i,k}}{(1-e^t)^k}, \tag{1.3}$$

where

$$\nu_{i,k} = (-1)^{k-1}\lambda_{i,k}.$$

Theorem 1.2 ([2,3]). *For $i \in \{0\} \cup \mathbb{N}$, we have*

$$\left(\frac{1}{1-e^{-t}}\right)^{(i)} = \sum_{k=1}^{i+1}\frac{\mu_{i,k}}{(1-e^{-t})^k}, \tag{1.4}$$

where

$$\mu_{i,k} = (-1)^{k+1}(k-1)! S(i+1,k).$$

As a consequence, the coefficients $a_{k,i-1}$ in Eq. (1.1) were calculated by a determinant:

Theorem 1.3 ([2,3]).

$$a_{k,i-1} = (-1)^{i^2+1} M_{k-i+1}(k,i), \tag{1.5}$$

where

$$M_j(k,i) = \begin{vmatrix} \frac{1}{(i-1)!}\binom{k}{i} & S(i+1,i) & \cdots & S(i+j-1,i) \\ \frac{1}{i!}\binom{k}{i+1} & S(i+1,i+1) & \cdots & S(i+j-1,i+1) \\ \frac{1}{(i+1)!}\binom{k}{i+2} & 0 & \cdots & S(i+j-1,i+2) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(i+j-2)!}\binom{k}{i+j-1} & 0 & \cdots & S(i+j-1,i+j-1) \end{vmatrix}. \tag{1.6}$$

In their paper Guo and Qi also gave a theorem which states that the functions $1/(e^t-1)^k$ can be expressed by linear combinations of the derivatives $(1/(e^t-1))^{(i)}$.

Theorem 1.4 ([2,3]). *For $k \in \mathbb{N}$, the identity*

$$\frac{1}{(e^t-1)^k} = \sum_{i=1}^{k} b_{k,i-1}\left(\frac{1}{e^t-1}\right)^{(i-1)} \tag{1.7}$$

is valid and the coefficients $b_{k,i-1}$ can be computed by

$$b_{k,i-1} = (-1)^{i-1} a_{k,i-1}. \tag{1.8}$$

It was pointed out by [2] that this theorem is essentially equivalent to Theorem 1.1.