Journal of Computational and Applied Mathematics 260 (2014) 218–228

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Application of the spectral pseudo-inversion to solving Hermitian systems of differential–algebraic equations

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ARTICLE INFO

*Article history:*
Received 17 October 2012
Received in revised form 18 July 2013

*Keywords:*
Differential–algebraic systems
Spectral pseudo-inversion
Spectral projection
Laguerre polynomials
Error bounds
RC-circuits

ABSTRACT

A new effective method for solving Hermitian differential–algebraic systems with constant coefficients is proposed and justified. It uses an explicit representation of the solution based on the spectral pseudo-inversion and an expansion of the matrix exponential via Laguerre polynomial series. An approximate solution is obtained by truncating the series. A new bound for the approximate solution error is derived. Results of numerical experiments with systems arising in interconnect analysis of VLSI circuits are presented and discussed.



## 1. Introduction

In the paper [1] a new generalized inversion (the spectral pseudo-inversion) for Hermitian matrices based on projections was introduced. When associated with a regular Hermitian matrix pencil, it can be expressed by a contour integral formula and can be used, in particular, to write down an explicit representation of the solutions of Hermitian linear differential–algebraic systems (DAS), to derive optimal bounds for the solution norms and to estimate the settling time [2]. In the paper [3] the spectral pseudo-inversion was extended to arbitrary square matrices. Such an extension includes as special cases known generalized inverses such as the Moore–Penrose and the Drazin inverses. It was used to show that the explicit representation of the solutions of DAS can be simplified when a well chosen expansion is used for the exponential function. An illustration was given with the expansion in Laguerre functions which are widely used in Krylov subspace reduction methods (see, e.g. [4–6]).

In this paper we adopt this approach to the following initial value problem for a Hermitian DAS with constant coefficients:

$$x(0) = 0, \quad E\frac{dx}{dt} = Ax + Bu, \tag{1}$$

where $x(t) \in \mathbf{C}^n$ is an unknown (state) vector function, $u(t) \in \mathbf{C}^p$ is a given (control) vector function ($p \ll n$), $B \in \mathbf{C}^{n\times p}$, $A \in \mathbf{C}^{n\times n}$, $E \in \mathbf{C}^{n\times n}$ and the matrices $A$ and $E$ are Hermitian with $A$ being negative definite and $E$ positive semidefinite. The

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http://dx.doi.org/10.1016/j.cam.2013.09.059

assumption on $A$ and $E$ implies that the pencil $\lambda E - A$ is non-singular and its finite eigenvalues, i.e., the roots of the equation $\det(\lambda E - A) = 0$, are real and negative.

Assuming that the function $u(t)$ and its derivative are bounded and piecewise continuous and $u(0) = 0$, we seek the solution of (1) in the form

$$x(t) = y(t) - A^{-1}Bu(t), \tag{2}$$

where $y(t)$ is a new unknown vector function. As we will see this change of variables facilitates the study and solution of (1) because $y(t)$ belongs to the deflating subspace of $\lambda E - A$ corresponding to the finite eigenvalues. Substituting (2) into (1) we find that $y(t)$ is the solution of the following initial value problem:

$$y(0) = 0, \quad E\frac{dy}{dt} = Ay + Eh \tag{3}$$

with

$$h = A^{-1}B\frac{du}{dt}. \tag{4}$$

Here and after we assume for determinacy that $u(t) = u(t-0)$ and, by definition,

$$\frac{du}{dt}(t) = \frac{du}{dt}(t-0)$$

at the points where the derivative is discontinuous.

Note that DAS arise in a wide variety of applications (see [7] and references therein). Initial value problems of the form (1) or (3) arise, for example, in interconnect analysis of VLSI circuits; see, e.g., [8, p.11] and Section 2 of the present paper. In Sections 3–4 we describe and justify a method for solving these problems. Details of its realization are discussed in Section 5 where, in particular, a new method for fast convolution with Laguerre polynomials is proposed.

The method proposed in the present paper can be viewed as a model order reduction in time domain. It belongs to the spectral methods that use non-local basis functions to approximate the solution. Such methods are evidently very efficient for problems whose solutions can be well-approximated with linear combinations of a small number of basis functions. Many problems arising in VLSI circuits analysis satisfy this condition and we consider some of them in Section 6 where we present results of numerical experiments. As basis functions, we use the Laguerre polynomials.

## 2. Application to RC-circuits

The modeling of an RC-circuit based on Ohm's and Kirchhoff's laws leads to a DAS of the following form:

$$E\frac{du_{in}}{dt} = Au_{in} + A_p u_p + E_p\frac{du_p}{dt} \tag{5}$$

where $u_p(t) \in \mathbf{R}^p$, $u_{in}(t) \in \mathbf{R}^n$ are vectors of voltage in the port and inner nodes, respectively, $E \in \mathbf{R}^{n\times n}$, $E_p \in \mathbf{R}^{n\times p}$ and $A \in \mathbf{R}^{n\times n}$, $A_p \in \mathbf{R}^{n\times p}$ are matrices representing the capacities and resistors, respectively. The matrices $E$ and $A$ are symmetric non-strictly diagonally dominant with nonnegative and negative diagonal elements, respectively [9]. Therefore, according to Gershgorin's theorem [10], they are respectively positive and negative semidefinites.

In addition, we will assume that each inner node can be reached from any port node by passing only via resistors. This property leads to the strict diagonal dominance of $A$, and therefore to its negative definiteness.

Note that for analyzing RC-circuits, the transfer function is often used instead of system (5) (see, e.g., [11]).

System (5) can be written in the form (1) with

$$x = u_{in}, \qquad B = [A_p, E_p], \qquad u = [u_p, du_p/dt]^T. \tag{6}$$

The obtained system satisfies all conditions formulated in Section 1 and can be reduced to the form (3). Let us show that some properties of the matrices in (5) allow us to transform this system to a system of the form (3) without increasing the dimension of control vector.

Denote by $P$ and $S$ some matrices of size $n \times p$ which satisfy

$$AP + A_p = 0 \tag{7}$$

and

$$ES + E_p = 0. \tag{8}$$

These equations are solvable for any RC-circuit. This property is well-known (see, e.g., [11,12]) and can be easily justified in terms of RC-circuits. Indeed, the $k$-th column of $P$ is the vector of voltages at the inner nodes for the circuit obtained from the initial one by removing all capacitors when the unit voltage is set at the $k$-th port node and zero in the others. Similarly,