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On the resolution power of Fourier extensions for oscillatory functions

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ABSTRACT

Functions that are smooth but non-periodic on a certain interval possess Fourier series that lack uniform convergence and suffer from the Gibbs phenomenon. However, they can be represented accurately by a Fourier series that is periodic on a larger interval. This is commonly called a Fourier extension. When constructed in a particular manner, Fourier extensions share many of the same features of a standard Fourier series. In particular, one can compute Fourier extensions which converge spectrally fast whenever the function is smooth, and geometrically fast if the function is analytic, much the same as the Fourier series of a smooth/analytic and periodic function.

With this in mind, the purpose of this paper is to describe, analyse and explain the observation that Fourier extensions, much like classical Fourier series, also have excellent resolution properties for representing oscillatory functions. The *resolution power*, or required number of degrees of freedom per wavelength, depends on a user-controlled parameter and, as we show, it varies between 2 and π. The former value is optimal and is achieved by classical Fourier series for periodic functions, for example. The latter value is the resolution power of algebraic polynomial approximations. Thus, Fourier extensions with an appropriate choice of parameter are eminently suitable for problems with moderate to high degrees of oscillation.

1. Introduction

In many physical problems, one encounters the phenomenon of oscillation. When approximating the solution to such a problem with a given numerical method, this naturally leads to the question of resolution power. That is, how many degrees of freedom are required in a given scheme to resolve such oscillations? Whilst it may be impossible to answer this question in general, important heuristic information about a given approximation scheme can be gained by restricting ones interest to certain simple classes of oscillatory functions (e.g. complex exponentials for problems in bounded intervals).

Resolution power represents an *a priori* measure of the efficiency of a numerical scheme for a particular class of problems. Schemes with low resolution power require more degrees of freedom, and hence increased computational cost, before the onset of convergence. Conversely, schemes with high resolution power capture oscillations with fewer degrees of freedom, resulting in decreased computational expense.

Consider the case of the unit interval $[-1, 1]$ (the primary subject of this paper). Here one typically studies the question of resolution via the complex exponentials

$$f(x) = \exp(\mathrm{i}\omega\pi x), \quad \omega \in \mathbb{R}. \tag{1}$$

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Nomenclature

Symbols

Symbol	Description
$f(x)$	Function being approximated (and extended)
r	Resolution constant of an approximation scheme
n	Trigonometric polynomial degree
T	Extension parameter
g_n	Fourier extension
$x \in [-1, 1]$	Physical domain variable
$y \in [c(T), 1]$	Transformed domain variable, $y = \cos \frac{\pi}{T} x$
$u \in [-1, 1]$	Transformed domain variable, $u = 2\frac{y-c(T)}{1-c(T)} - 1$
$\mathbb{T}$	The torus $[-T, T)$
$E(T)$	Fourier extension convergence constant $\cot^2(\frac{\pi}{4T})$
$r(T)$	Fourier extension resolution constant
$c(T)$	$\cos(\pi/T)$
G_n	Space of $2T$-periodic functions
C_n, S_n	Spaces spanned by $\cos \frac{\pi}{T} kx$ and $\sin \frac{\pi}{T}(k+1)x$, $k = 0, \dots, n$
$e(\rho)$	Bernstein ellipse
$A, \tilde{A}$	Matrices of the continuous and discrete Fourier extensions
x_i	Symmetric mapped Chebyshev nodes
$W(x)$	The weight function $\frac{2\pi}{T} \cos \frac{\pi}{2T} x / \sqrt{\cos \frac{\pi}{T} x - \cos \frac{\pi}{T}}$
κ	Condition number
ϵ	Machine precision
ω	Frequency of oscillation

To this end, let $\phi_n(f)$, $n = 1, 2, \dots$ be a sequence of approximations of the function $f(x) = \exp(\mathrm{i}\pi\omega x)$ which converges to f as $n \to \infty$ (here n is the number of degrees of freedom in the approximation $\phi_n(f)$). For $0 < \delta < 1$, let $n(\delta, \omega)$ be the minimal n such that

$$\|f - \phi_n(f)\|_{L^2_{[-1,1]}} < 2\delta.$$

We now define the *resolution constant* r of the approximation scheme $\{\phi_n\}$ as

$$r = \limsup_{\delta \to 1^-} \lim_{\omega \to \infty} \frac{n(\delta, \omega)}{\omega}.$$

Note that r need not be well defined for an arbitrary scheme $\{\phi_n\}$ (for example, if $n(\delta, \omega)$ were to scale superlinearly in ω). However, for all schemes encountered in this paper, this will be the case.

Loosely speaking, the resolution constant r corresponds to the required number of degrees of freedom per wavelength to capture oscillatory behaviour; a common concept in the literature on oscillatory problems [1,2]. In particular, we say that a given scheme has high (respectively low) resolution power if it has small (large) resolution constant. It is also worth noting that, in many schemes of interest, the approximation $\phi_n(f)$ is based on a collocation at a particular set of n nodes in $[-1, 1]$. In this circumstance, the resolution constant r is equivalent to the number of *points per wavelength* required to resolve an oscillatory wave (for further details, see Section 1.3).

We remark in passing that the reason for defining r as two limits is as follows. First, we take the limit in ω since our interest lies in the oscillatory regime $\omega \gg 1$. This is not strictly necessary, but makes the analysis easier since we may consider asymptotic expansions in $\omega \to \infty$. Second, the reason for the limit $\delta \to 1^-$ is so that r measures the point at which the approximation $\phi_n(f)$ begins to converge, i.e. the minimal n possible to resolve f. It therefore provides a quick and simple quantitative means to compare different methods.

With little doubt, the approximation of a smooth, periodic function via its truncated Fourier series is one of the most effective numerical methods known. Fourier series, when computed via the FFT, lead to highly efficient, stable methods for the numerical solution of a large range of problems (in particular, PDEs with periodic boundary conditions). A simple argument leads to a resolution constant of $r = 2$ in this case (for periodic oscillations), with geometric convergence occurring once the number of Fourier coefficients exceeds 2ω.

However, the situation is altered completely once periodicity is lost. The slow pointwise convergence of the Fourier series of a nonperiodic function, as well as the presence of $\mathcal{O}(1)$ Gibbs oscillations near the domain boundaries, means that nonperiodic oscillations cannot be resolved by such an approximation. A standard alternative is to approximate with a sequence of orthogonal polynomials (e.g. Chebyshev polynomials). Such approximations possess geometric convergence,