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Asymptotic mean-square stability of two-step Maruyama schemes for stochastic differential equations

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ABSTRACT

The mean-square stability for two-step schemes applied to scalar stochastic differential equations is studied. Necessary and sufficient conditions in terms of the parameters of the schemes guaranteeing their MS-stability are derived. Particular members of the studied family are considered, their stability regions are plotted and compared with the stability region of the linear test equation. It is proved that the stochastic two-step BDF scheme is unconditionally MS-stable. Numerical experiments that confirm the theoretical results are shown.

1. Introduction

In this paper we consider scalar stochastic differential equations (SDEs) of Itô type of the form

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dW_t; \qquad X_{t_0} = x_0, \tag{1}$$

where x_0 is a constant, W is the standard one-dimensional Wiener process, and the coefficients $f = f(t, x)$ and $g = g(t, x)$ with $t \in [t_0, T]$, $x \in \mathbb{R}$, satisfy the assumptions of the existence and uniqueness theorem, see [1], and are continuous with respect to t. There exists an evident interest in the development of stochastic numerical methods to solve (1) due to the fact that analytic solutions of SDEs are, in general, not available.

In this paper we deal with stochastic linear two-step schemes, also called *two-step Maruyama schemes*, which are a direct extension of the homonymous deterministic class. They give approximations $\{X_n\}$ to the solution of (1) of the form

$$\begin{aligned} X_{n+1} + \alpha_1 X_n + \alpha_0 X_{n-1} &= \Delta\left(\beta_2 f(t_{n+1}, X_{n+1}) + \beta_1 f(t_n, X_n) + \beta_0 f(t_{n-1}, X_{n-1})\right) \\ &\quad + \gamma_1\, g(t_n, X_n)\Delta W_n + \gamma_0\, g(t_{n-1}, X_{n-1})\Delta W_{n-1} \end{aligned} \tag{2}$$

where $\Delta > 0$ is the constant step size, ΔW_n are independent $N(0, \Delta)$ Gaussian random variables and $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1$ are real parameters; notice that the coefficient $\alpha_2 = 1$ of X_{n+1} has been normalized. Since (2) is an extension of its counterpart linear two-step scheme for deterministic differential equations, take $g \equiv 0$ in (1) and (2), the "deterministic" coefficients $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$ are supposed to fulfill conditions that ensure the convergence of the deterministic counterpart method:

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Table 1
Coefficients of two-step Maruyama schemes for SDEs.

Method	α_2	α_1	α_0	β_2	β_1	β_0	γ_1	γ_0
Adams–Bashforth Maruyama	1	-1	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	0
Adams–Moulton Maruyama	1	-1	0	$\frac{5}{12}$	$\frac{8}{12}$	$\frac{-1}{12}$	1	0
Midpoint Maruyama rule	1	0	-1	0	2	0	1	1
Simpson Maruyama rule	1	0	-1	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{3}$	1	1
BDF Maruyama	1	$\frac{-4}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	0	1	$\frac{-1}{3}$

(i) Dahlquist's root condition, i.e., the polynomial $\rho(z) = z^2 + \alpha_1 z + \alpha_0$ has all its zeros in the unit disk $|z| \leq 1$ without multiple zeros on the border $|z| = 1$.
(ii) Deterministic consistency conditions:

$$1 + \alpha_1 + \alpha_0 = 0, \qquad 2 + \alpha_1 = \beta_0 + \beta_1 + \beta_2.$$

In addition, stochastic consistency conditions, see [2],

$$\gamma_1 = 1, \qquad \gamma_0 = 1 + \alpha_1 \tag{3}$$

guaranteeing that the stochastic scheme is mean-square convergent are imposed. Notice that conditions (3) determine the values of diffusion parameters γ_0, γ_1 once α_1 is fixed. Then each deterministic linear two-step convergent method has exactly one stochastic version in the two-step Maruyama class. Due to this fact, the stochastic schemes (2) will be named adding the terms stochastic or Maruyama to their deterministic denomination. Particular schemes of this family are shown in Table 1, where their parameters are displayed.

As in the deterministic case, an important feature of numerical schemes is their ability to produce numerical solutions that preserve qualitative properties of the exact solution of the equation to which they are applied. Here we are interested in linear MS-stability, i.e., how two-step Maruyama schemes (2) reproduce asymptotical mean-square stability properties of the solution of the multiplicative noise linear test equation

$$dX_t = \lambda X_t dt + \mu X_t\, dW_t \tag{4}$$

with real parameters $\lambda,\ \mu$. The notion of MS-stability for stochastic numerical schemes has been proved to be a successful extension of A-stability for scalar one-step methods, see, e.g., [3–9] and, recently, for one-step methods applied to multidimensional stochastic differential equations, see [2,10,11]. But a barrier has been observed to extend the concept of MS-stability to multi-step schemes and few attempts at tackling this problem can be found, see [10,12,13]. Based on the construction of appropriate Lyapunov functionals, in [12] the authors give general sufficient conditions for mean-square stability of linear two-step Maruyama schemes and claim that it would be desirable to obtain necessary conditions, even when these are not identical to the sufficient conditions. In the present paper, reducing the scalar two-step MS-stability analysis to the study of the stability of a linear multidimensional system governed by a constant matrix, we derive necessary and sufficient conditions on the parameters of two-step Maruyama schemes (2) for their MS-stability. As a further outcome, we see how the results of Shaikhet [14,15], obtained also from a Lyapunov functional and devoted to a special class of stochastic linear difference equations, are included in our general framework.

The paper is organized as follows: In Section 2 the main concepts and well-known results on MS-stability theory are recalled; at the end of this section we see how the intended MS-stability analysis for two-step Maruyama schemes can be stated as the MS-stability study for a general second order stochastic difference equation. In Section 3, the problem is reduced to the stability analysis of a linear system recurrence, and, then, characterized in terms of the spectral radius of the associated matrix. Based on Schur–Cohn and Jury–Marden lemmas, the characterization is also given in terms of the coefficients of the matrix. In Section 4 the obtained necessary and sufficient conditions for the MS-stability are translated to the two-step Maruyama scheme case. In particular, regions of MS-stability are determined and plotted for the schemes that appear in Table 1. Finally, Section 5 is devoted to some numerical experiments that confirm the theoretical results.

2. Basic concepts and preliminary results

2.1. Asymptotical MS-stability

Consider the SDE (1) and assume that $f(t, 0) = 0$ and $g(t, 0) = 0$ for $t \geq t_0$. Notice that this implies that the process $X_t \equiv 0$, called the *equilibrium position*, is the (unique) solution of (1) with $x_0 = 0$.

Definition 1 (*[1,16]*)**.** The equilibrium position is said to be stable in mean square if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{t_0 \leq t < \infty} E|X_t(x_0)|^2 \leq \varepsilon \quad \text{for } |x_0| \leq \delta.$$