Journal of Computational and Applied Mathematics 260 (2014) 356–363

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A mixed finite element method for nearly incompressible elasticity and Stokes equations using primal and dual meshes with quadrilateral and hexahedral grids

Bishnu P. Lamichhane *

School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia

HIGHLIGHTS

- Optimal scheme is given for nearly incompressible elasticity and Stokes equations.
- Primal mesh for the displacement and dual mesh for the pressure.
- The earlier results are extended to quadrilateral and hexahedral meshes.
- Optimal error estimates are proved.
- Displacement-based formulation is derived.

ARTICLE INFO

Article history:
Received 29 April 2013

MSC:
65N30
65N15
74B10

Keywords:
Mixed finite elements
Nearly incompressible elasticity
Primal and dual meshes
Stokes equations
Inf–sup condition

ABSTRACT

We consider a mixed finite element method for approximating the solution of nearly incompressible elasticity and Stokes equations. The finite element method is based on quadrilateral and hexahedral triangulation using primal and dual meshes. We use the standard bilinear and trilinear finite element space enriched with element-wise defined bubble functions with respect to the primal mesh for the displacement or velocity, whereas the pressure space is discretized by using a piecewise constant finite element space with respect to the dual mesh.

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1. Introduction

Although there are many mixed finite element methods for nearly incompressible elasticity and Stokes equations leading to an optimal convergence, the search for simple, efficient and optimal finite element schemes is still an active area of research. In this article we present a mixed finite element method for nearly incompressible elasticity and Stokes equations using quadrilateral and hexahedral meshes. The displacement or velocity field is discretized by using the standard bilinear or trilinear finite element space enriched with element-wise defined bubble functions, whereas the pressure space is discretized by the piecewise constant finite element space based on a dual mesh. Such a finite element space for the simplicial mesh is presented in [1], where the inf–sup condition is proved by using the fact that the mini finite element [2]

* Tel.: +61 422437170.
E-mail addresses: blamichha@gmail.com, Bishnu.Lamichhane@anu.edu.au.

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http://dx.doi.org/10.1016/j.cam.2013.09.056

satisfies the inf–sup condition. Note that the mini finite element [2] consists of the linear finite element space enriched with element-wise defined bubble functions for the displacement or velocity and the linear finite element space for the pressure space. The enrichment of the displacement or velocity field increases one vector degree of freedom per element. A main hindrance to extend this approach to the case of quadrilateral and hexahedral meshes is that the displacement or velocity space should be enriched by more than a single bubble function to obtain the inf–sup condition [3]. In this article we show that a similar discretization scheme can be applied to quadrilateral and hexahedral meshes. We prove that if the pressure space is discretized by using the piecewise constant function space with respect to the dual mesh, it is sufficient to enrich the standard bilinear and trilinear finite element space with a single bubble function per element.

2. The boundary value problem of linear elasticity

We introduce the boundary value problem of linear elasticity in this section. In particular, we present the standard weak formulation and a mixed formulation of a linear elastic problem. We consider a homogeneous isotropic linear elastic material body occupying a bounded domain $\Omega \subset \mathbb{R}^d$, $d = \{2, 3\}$, with Lipschitz boundary Γ. For a prescribed body force $\boldsymbol{f} \in [L^2(\Omega)]^d$, the governing equilibrium equation in Ω reads

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f}, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The stress tensor $\boldsymbol{\sigma}$ is defined as a function of the displacement $\boldsymbol{u}$ by the Saint-Venant–Kirchhoff constitutive law

$$\boldsymbol{\sigma} = \frac{1}{2}\mathcal{C}(\nabla \boldsymbol{u} + [\nabla \boldsymbol{u}]^t), \tag{2.2}$$

where $\mathcal{C}$ is the fourth-order elasticity tensor. The action of the elasticity tensor $\mathcal{C}$ on a tensor $\boldsymbol{d}$ is defined as

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d} := \lambda(\operatorname{tr} \boldsymbol{d})\mathbf{1} + 2\mu\, \boldsymbol{d}. \tag{2.3}$$

Here, $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and they are assumed to be positive. For simplicity of exposition we assume that the displacement or velocity satisfies homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma. \tag{2.4}$$

However, the approach works also for mixed boundary conditions.

Standard weak formulation.

Let $L^2(\Omega)$ be the set of square-integrable functions defined on Ω, where the inner product and norm on this space is denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, respectively. The Sobolev space $H^1(\Omega)$ is defined in terms of the space $L^2(\Omega)$ as

$$H^1(\Omega) = \left\{u \in L^2(\Omega),\ \nabla u \in [L^2(\Omega)]^d\right\},$$

and $H_0^1(\Omega) \subset H^1(\Omega)$, where a function in $H_0^1(\Omega)$ vanishes on the boundary in the sense of traces. The space $L_0^2(\Omega)$ is the subset of $L^2(\Omega)$ defined as

$$L_0^2(\Omega) = \left\{p \in L_0^2(\Omega) :\ \int_\Omega p\, d\boldsymbol{x} = 0\right\}.$$

To write the weak or variational formulation of the boundary value problem, we introduce the space $\boldsymbol{V} := [H_0^1(\Omega)]^d$ of displacement or velocity with inner product $(\cdot,\cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way; that is, $(\boldsymbol{u}, \boldsymbol{v})_1 := \sum_{i=1}^d (u_i, v_i)_1$, with the norm being induced by this inner product.

We define the bilinear form $A(\cdot,\cdot)$ and the linear functional $\ell(\cdot)$ by

$$A : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}, \quad A(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\boldsymbol{x},$$

$$\ell : V \to \mathbb{R}, \quad \ell(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x}.$$

Then the standard weak form of the linear elasticity problem is as follows: given $\ell \in \boldsymbol{V}'$, find $\boldsymbol{u} \in \boldsymbol{V}$ that satisfies

$$A(\boldsymbol{u}, \boldsymbol{v}) = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{2.5}$$

The assumptions on $\mathcal{C}$ guarantee that $A(\cdot,\cdot)$ is symmetric, continuous, and $\boldsymbol{V}$-elliptic. Hence by using standard arguments it can be shown that (2.5) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}$. Furthermore, if the domain Ω is convex with polygonal or polyhedral boundary, $\boldsymbol{u} \in [H^2(\Omega)]^d \cap \boldsymbol{V}$, and there exists a constant C independent of λ such that [4–6]

$$\|\boldsymbol{u}\|_2 + \lambda\|\operatorname{div} \boldsymbol{u}\|_1 \le C\, \|\boldsymbol{f}\|_0. \tag{2.6}$$

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