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Application of a conservative, generalized multiscale finite element method to flow models

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ABSTRACT

In this paper, we propose a method for the construction of locally conservative flux fields from Generalized Multiscale Finite Element Method (GMsFEM) pressure solutions. The flux values are obtained from an element-based postprocessing procedure in which an independent set of 4×4 linear systems need to be solved. To test the performance of the method we consider two heterogeneous permeability coefficients and couple the resulting fluxes to a two-phase flow model. The increase in accuracy associated with the computation of the GMsFEM pressure solutions is inherited by the postprocessed flux fields and saturation solutions, and is closely correlated to the size of the reduced-order systems. In particular, the addition of more basis functions to the enriched coarse space yields solutions that more accurately capture the behavior of the fine scale model. A number of numerical examples are offered to validate the performance of the method.

1. Introduction

Many physical processes in science and engineering are described by partial differential equations whose coefficients vary over many length scales. Typical examples may include subsurface flows where the permeability of the porous medium is represented by a high-contrast, heterogeneous coefficient. In recent decades, multiscale methods have been introduced as an effective tool for treating these types of problems [1–6]. An important component of this class of methods is the independent construction of a set of multiscale basis functions that span a solution space that is tied to a coarse grid, i.e., one whose discretization parameter is much larger than the characteristic scale of the heterogeneous coefficient. In particular, once a precomputed set of basis functions is available, a specified global coupling mechanism may be used in order to obtain the associated coarse scale solution. As the fine scale information is embedded into the basis functions, a coarse grid solution inherits the fine scale effects of the underlying system. In other words, the multiscale basis functions offer a direct method of projecting a coarse solution to the fine grid. The present paper considers a class of multiscale methods that will be used to effectively solve elliptic pressure equations that appear in a two-phase flow model.

While standard multiscale methods have proven effective for a variety of applications (see, e.g., [7,4,6]), we employ a more recent framework in which the coarse space may be systematically enriched so that the approximate solution sought in it converges to the fine grid solution. The enrichment procedure hinges on the construction of localized spectral problems, where dominant eigenfunctions are used in the construction of the enriched space [8,9]. This type of spectral enrichment allows for the number basis functions (and the size of the coarse space) to be flexibly chosen such that a desired level

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of numerical accuracy may be achieved. This framework, which is coined as the Generalized Multiscale Finite Element Method (GMsFEM), incorporates the enriched solution space into a global formulation in order to obtain approximate pressure solutions. The global formulation considered here is the continuous Galerkin (CG) method. Other choices, such as the discontinuous Galerkin method are possible for the global coupling [10]. To the best of our knowledge, it is still an open problem for some global couplings such as the finite volume element method.

An advantage of employing a continuous Galerkin multiscale formulation is the relative ease of implementation and resemblance to standard finite element variational formulation. However, a well known limitation of CG is that the resulting solution does not satisfy local conservation. In particular, in the cases when it is necessary to couple the resulting fluxes to a transport equation, local conservation is required. While finite volume-type methods, mixed methods, and discontinuous Galerkin methods typically guarantee conservation [1,11,7], the respective formulations yield systems that are more delicate (and sometimes larger) than the CG counterpart. Furthermore, to the best of our knowledge the blend of enrichment techniques with multiscale methods are still at its infancy as there has not been any attempt to carry out the formulation using other than continuous Galerkin formulation (see, e.g., [12] for a recent development using discontinuous Galerkin method). As a result, we consider the alternative of postprocessing a global CG solution in order to obtain the desired conservation. Numerous methods have been proposed in order to postprocess finite element solutions to obtain conservative fluxes (see, e.g., [13–16]), however, in this paper we generalize the procedure offered in [17] in which the authors perform a global solve and subsequent element-based computations to achieve conservation.

In this paper, we propose a technique that provides flux conservation in the context of GMsFEM. For rectangular finite elements, our method hinges on a postprocessing technique in which independent 4×4 systems of equations are solved on each coarse element to obtain the conservative fluxes. We note that similar derivation can be accomplished for triangular finite elements that yields an independent 3×3 system of equations. While the postprocessing procedure yields conservative fluxes on the coarse scale (which might suffice for some target applications), we also employ an independent downscaling procedure to construct a conservative flux field on the underlying fine grid. We note that coarse scale conservative discontinuous Galerkin GMsFEM formulations have been used in (see, e.g., [12]), yet emphasize that the method proposed in this paper requires no modification to the original CG formulation and allows for the fluxes to be computed on the fine grid. Furthermore, to our knowledge, conservative GMsFEM-type methods have not yet been incorporated for solving multi-phase flow models in the existing literature. To test the performance of the proposed method we solve a standard two-phase flow model using distinct cases of high-contrast permeability coefficients. In all cases, an increase in the dimension of the coarse solution space yields solutions that are shown to more accurately capture the behavior of the fine scale. In particular, the error decline of the elliptic solution (which has been rigorously analyzed in [8]), is directly inherited by the resulting flux values and two-phase saturation solutions.

The rest of the paper is organized as follows. In Section 2 we introduce a standard two-phase model along with a description of the operator splitting technique that is used for solving the model. In Section 3 we describe the Generalized Multiscale Finite Element Method (GMsFEM), and follow the construction by introducing the procedure for the computation of postprocessed, conservative flux quantities in Section 4. A variety of numerical tests are offered in Section 5 in order to validate the performance the proposed method. To finish the paper we offer some concluding remarks in Section 6.

2. Model problem

2.1. Two-phase model

We consider a heterogeneous oil reservoir which is confined in a domain Ω. The reservoir is equipped with an injection well, from which water is discharged to displace the trapped oil towards the production wells, situated on the perimeter of the domain. The dynamics of the movement of the fluids in the reservoir are categorized as an immiscible two-phase system with water and oil (denoted by w and o, respectively) that is incompressible. Capillary pressure is not included in the model. Further simplifying assumptions that we use are a gravity-free environment and that the two fluids fill the pore space. Then, the Darcy's law combined with a statement of conservation of mass allows us to write the governing equations of the flow as

$$\nabla \cdot \boldsymbol{v} = q, \quad \text{where } \boldsymbol{v} = -\lambda(S)k(\boldsymbol{x})\nabla p, \tag{2.1}$$

and

$$\frac{\partial S}{\partial t} + \nabla \cdot (f(S)\boldsymbol{v}) = q_w, \tag{2.2}$$

where $\boldsymbol{v}$ is the Darcy velocity, S is the water saturation, and k is the permeability coefficient. The total mobility $\lambda(S)$ and the flux function $f(S)$ are respectively given by:

$$\lambda(S) = \frac{k_{rw}(S)}{\mu_w} + \frac{k_{ro}(S)}{\mu_o}, \qquad f(S) = \frac{k_{rw}(S)/\mu_w}{\lambda(S)}, \tag{2.3}$$

where $k_{rj}, j = w, o$, is the relative permeability of the phase j.