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Superfast solution of linear convolutional Volterra equations using QTT approximation

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ABSTRACT

We address a linear fractional differential equation and develop effective solution methods using algorithms for the inversion of triangular Toeplitz matrices and the recently proposed QTT format. The inverses of such matrices can be computed by the divide and conquer and modified Bini's algorithms, for which we present the versions with the QTT approximation. We also present an efficient formula for the shift of vectors given in QTT format, which is used in the divide and conquer algorithm. As a result, we reduce the complexity of inversion from the fast Fourier level $\mathcal{O}(n \log n)$ to the speed of superfast Fourier transform, i.e., $\mathcal{O}(\log^2 n)$. The results of the paper are illustrated by numerical examples.

1. Introduction

Equations involving derivatives of fractional order are of great importance, due to their role in mathematical models applied in mechanics, biochemistry, electrical engineering, medicine, etc., see [1–3]. In this paper we present a superfast algorithm for the numerical solution of the linear equation

$$D_*^\alpha y(t) = F(t, y(t)) = my(t) + f(t), \quad 0 \le t \le T, \ y(0) = y_0, \tag{1}$$

where $0 < \alpha < 1$ is the order of the fractional operator, $m \in \mathbb{R}$ is a constant referred to as *mass*, and $f(t)$ is a sufficiently well-behaved *forcing* term. For $\alpha = 1/2$ this equation is a scalar version of the Bagley–Torvik equation [4], which is used in the modeling of viscoelastic materials. The definitions of Caputo derivative D_*^α can be found in many sources, e.g. [5,6], and are presented in the appendix for the convenience.

The classical result of Diethelm [5, Lemma 6.2] allows us to rewrite (1) in the form

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (my(s) + f(s))\, ds, \tag{2}$$

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where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ is the gamma function. Eq. (2) is the weakly singular convolutional Volterra equation of the second kind with the Abel-type kernel. Volterra equations of the second kind are well-studied and are proven to have a unique continuous solution for $0 \le t \le T$, see, e.g. [7, Theorem 3.2]. The solution is asymptotically stable if $m < 0$ (see [8]) which we will always assume in this paper.

For certain forcing terms, the solution of (2) can be found using series methods. In a general framework, we can discretize (2) using a collocation or Galerkin method and numerically solve the resulted linear system. This matrix approach to fractional calculus was brilliantly presented by I. Podlubny in [9]. In this paper we consider the collocation method and assume that $y(t)$ is approximated by a piecewise-linear function on a uniform grid $t_j = jh, j = 0, \dots, n$, where $h = T/n$. The stability of collocation methods for fractional equations was studied in [10,11] and an error analysis can be found in [12]. The discretized equation is the following

$$y_j = y_0 + \frac{h^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{j} w_{j,k}(my_k + f_k), \quad j = 1, \dots, n,$$

where $y_j = y(t_j), f_k = f(t_k)$ and $w_{j,k}$ are quadrature weights, defined by integration of piecewise-linear basis functions with Abel-type kernel, i.e.,

$$w_{j,k} = \frac{1}{\alpha(\alpha+1)} \begin{cases} (j-1)^{\alpha+1} - (j-\alpha-1)j^\alpha, & k = 0, \\ (j-k-1)^{\alpha+1} - 2(j-k)^{\alpha+1} + (j-k+1)^{\alpha+1}, & 1 \le k < j, \\ 1, & k = j. \end{cases}$$

Finally, we obtain the linear system $Ay = b$ with triangular Toeplitz matrix and the right-hand side defined as follows,

$$\sum_{k=1}^{j} a_{j-k} y_k = b_j, \quad j = 1, \dots, n, \tag{3}$$

$$a_p = \begin{cases} 1 - \gamma m, & p = 0, \\ -\gamma m \left((p-1)^{\alpha+1} - 2p^{\alpha+1} + (p+1)^{\alpha+1}\right), & p > 0, \end{cases}$$

$$b_j = y_0 + \gamma \left(\sum_{k=1}^{j} w_{j,k} f_k + w_{j,0}(my_0 + f_0) \right),$$

where $\gamma = h^\alpha / \Gamma(\alpha+2)$.

The numerical scheme we use is analogous to the fractional Adams method proposed in [12] for a general (e.g. nonlinear) function $F(t, y(t))$. The method is developed as a generalization of the Adams–Bashforth–Moulton scheme from the classical numerical analysis of ordinary differential equations and a detailed error analysis is provided. The complexity of the fractional Adams method in the nonlinear case is $\mathcal{O}(n^2)$. To reduce this complexity, we can take into account the decay speed of the Abel kernel $k(s) = s^{\alpha-1}$ of the integral in (2). The so-called *fixed memory principle* [6,13] and more accurate *nested mesh method* [14,15] are based on truncation and approximation of the tail of the integral (2), respectively, and have almost linear complexity w.r.t. n. We revise these methods in Section 2.

For linear $F(t, y(t))$, the problem writes as the linear system (3), which can be solved using well-developed algorithms for the inversion of triangular Toeplitz matrices, or triangular strip matrices, as they are referred to in [9]. These methods are recalled in Section 3, and have the asymptotic complexity of the fast Fourier transform (FFT) algorithm, which is $\mathcal{O}(n \log n)$.

Recently, a superfast Fourier transform algorithm was proposed in [16], based on the approximation of vectors in the *quantized tensor train* (QTT) format [17,18]. The method can be considered as a classical model of quantum superfast Fourier transform algorithm [19], and has a square-logarithmical complexity $\mathcal{O}(\log^2 n)$ for a certain class of vectors, for which such a model is efficient. This class of vectors is partially established in [20] and includes, for example, vectors with sparse Fourier image. The numerical experiments provided in Section 4 show that the Abel kernel $t(s) = s^{1-\alpha}$ is efficiently approximated by the QTT format for all $0 < \alpha < 1$ with accuracy up to the machine threshold. Based on this observation, we propose the superfast inversion algorithm for the triangular Toeplitz matrix (3), using the QTT approximation.

The numerical experiments provided in Section 5 justify the accuracy and sublinear complexity of the method proposed.

2. Numerical method with logarithmic memory

In [6,13] the author describes an approach to the numerical integration involved in solving a fractional problem whereby the first part (or tail) of the integral is ignored (i.e. assuming the value of the integral over this region is negligible) and so the memory of the system is truncated at some point. The error introduced via this process is described in [6] for Riemann–Liouville fractional derivatives. In [14] the authors consider the error that is introduced when this approach is applied to problems expressed with respect to the Caputo fractional derivative. The authors show that by introducing a finite memory of fixed length T for the Caputo derivative we introduce an error of the form

$$E = \left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t-T} \frac{y'(s)}{(t-s)^\alpha} ds \right|. \tag{4}$$