Journal of Computational and Applied Mathematics 260 (2014) 470–481

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On a discrete-time risk model with general income and time-dependent claims

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ARTICLE INFO

Article history:
Received 19 July 2013
Received in revised form 12 October 2013

MSC:
62P05
91B30

Keywords:
General premium rate
Time-dependent claim
Expected discounted penalty function
Defective renewal equation
Generating function

ABSTRACT

We consider a discrete-time risk model with general premium rate and time-dependent claim sizes, in which the interclaim time has an impact on the subsequent claim size. By studying the roots of Lundberg's generalized equation, we first obtain an analytical expression for the generating function of the expected discounted penalty function. Then it is shown that the expected discounted penalty function satisfies a defective renewal equation. Moreover, a closed-form expression for the generating function of the time to ruin is obtained when the claim sizes have discrete K_m distributions. Numerical examples are also given to illustrate the applicability of the results obtained.

1. Introduction

In risk theory, most results regarding the discrete-time risk models are based on a common assumption that interclaim arrivals and claim sizes are independent. See e.g. [1–3] in the framework of the compound binomial risk model, and [4,5] in the framework of the discrete-time renewal risk model. However, the independence assumption can be inappropriate and unrealistic in practical contexts. In the last decade, risk models with dependence between interclaim times and individual claim sizes have drawn many authors' attention. Among others, Yuen and Guo [6] investigate a risk process with delayed claims in the compound binomial risk model. Cossette et al. [7] consider the so-called compound Markov binomial model which introduces dependency between claim occurrences. The discrete-time renewal risk model with an arbitrary dependence structure is considered by Woo [8].

For all of the above-mentioned risk models, it is explicitly assumed that the premium received per period is one. For continuous risk models, the unit premium rate can be extended to an arbitrary constant premium rate by appropriately rescaling the time as well as the claim sizes. Unfortunately, such reasoning does not hold for fully discrete settings. Thus, some actuarial science practitioners have analyzed discrete-time risk models with a general premium rate c ($c \in \mathbb{N}^+$), see e.g. [9,10].

In the present paper, we consider a fully discrete risk model, which combines an arbitrary premium rate and a threshold dependence structure. Although such a combination complicates the study of ruin measures, it makes our model more

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http://dx.doi.org/10.1016/j.cam.2013.10.031

realistic and adjustable. The surplus process is described as follows

$$U_k = u + ck - \sum_{i=1}^{N_k} X_i, \tag{1.1}$$

where $U_0 = u \in \mathbb{N}$ is the initial capital allocated to the portfolio and $c \in \mathbb{N}^+$ is the premium received per time unit. Suppose that the number of claims $\{N_k, k \in \mathbb{N}\}$ is a binomial process. More precisely, the interclaim times $\{W_i, i \in \mathbb{N}^+\}$ are a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s) having probability mass function (p.m.f.) $f_W(n) = (1-q)\,q^{n-1}$, $n \in \mathbb{N}^+$. The individual claim amounts $\{X_i, i \in \mathbb{N}^+\}$ are assumed to be i.i.d. r.v.'s with support $\mathbb{N}^+$ and p.m.f. f_X.

It is assumed that the pairs (W_i, X_i), $i \in \mathbb{N}^+$ form a sequence of i.i.d. random vectors, in which the components W_i and X_i are no longer independent. A specific dependence structure between W_i and X_i is defined as follows. Suppose that $\{V_i, i \in \mathbb{N}^+\}$ are a sequence of i.i.d. threshold r.v.'s with p.m.f. $f_V(n) = (1-\theta)\,\theta^{n-1}$ for $n \in \mathbb{N}^+$, which are independent of all the other r.v.'s associated with (1.1). If W_i is smaller than (equal to or larger than) V_i, then the subsequent claim size X_i has p.m.f. f_1 (f_2). Under these assumptions, the (conditional) p.m.f. of $X|W$ is a special mixture of f_1 and f_2 (with respective expectations μ_1 and μ_2), i.e.

$$f_{X|W}(n) = \theta^W f_1(n) + \left(1-\theta^W\right) f_2(n), \quad n \in \mathbb{N}^+. \tag{1.2}$$

We remark that the dependence structure (1.2) makes sense in reality. More precisely, if the time elapsed since the last catastrophe is too long, people's memory of it will decline and the current catastrophe will probably cause a sizable loss, vice versa. Furthermore, this kind of threshold dependence structure has received considerable interests in a continuous time framework [11–15].

The time to ruin is defined as $T = \min\left\{k \in \mathbb{N}^+, U_k < 0\right\}$ with $T = \infty$ if ruin does not occur. When ruin occurs, U_{T-1} is the surplus one period prior to ruin and $|U_T|$ is the deficit at ruin. For $v \in (0, 1]$, the expected discounted penalty function is defined as

$$\Phi_v(u) = \mathrm{E}\left[v^T w\left(U_{T-1}, |U_T|\right)\mathbf{1}_{\{T<\infty\}} \mid U_0 = u\right], \tag{1.3}$$

where $w : \mathbb{N} \times \mathbb{N}^+ \to \mathbb{N}$ is a penalty function and $\mathbf{1}_{\{A\}}$ is the indicator function of an event A. Also, we consider some special cases of (1.3) with successively simplified the penalty functions. If $w(n_1, n_2) = 1$ for $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}^+$, we get the generating function of the time to ruin, i.e.

$$\psi_v(u) = \mathrm{E}\left[v^T \mathbf{1}_{\{T<\infty\}} \mid U_0 = u\right]. \tag{1.4}$$

Obviously, when $v = 1$, (1.4) reduces to the probability of ruin $\psi_1(u) = \Pr\{T < \infty \mid U_0 = u\}$. If $v = 1$ and $w(n_1, n_2) = \mathbf{1}_{\{n_2 \le x\}}$, $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}^+$, we get the (defective) distribution function of the deficit at ruin, i.e.

$$\varphi(u, x) = \Pr\left\{T < \infty, |U_T| \le x \mid U_0 = u\right\}. \tag{1.5}$$

To ensure that ruin will not almost surely occur, we always assume $\mathrm{E}(cW - X) > 0$. From (1.2), the resulting marginal distribution of X is

$$f_X(n) = \frac{(1-q)\,\theta}{1-q\theta} f_1(n) + \frac{1-\theta}{1-q\theta} f_2(n), \quad n \in \mathbb{N}^+,$$

which yields a net profit condition as follows

$$\frac{(1-q)\,\theta\mu_1 + (1-\theta)\,\mu_2}{1-q\theta} < \frac{c}{1-q}. \tag{1.6}$$

The rest of the paper is structured as follows. In Section 2, we analyze Lundberg's generalized equation and its roots. The generating function and the defective renewal equation for the expected discounted penalty function are derived in Sections 3 and 4, respectively. In Section 5, we obtain the explicit expression for the generating function of the time to ruin for a large class of claim size distributions. Finally, numerical illustrations are given in Section 6.

2. Lundberg's generalized equation

In this section, we find some useful martingales to derive Lundberg's generalized equation, which is a major subject of study for the ruin measures.

Let $T_0 = 0$ and $T_k = \sum_{i=1}^k W_i$, $k \in \mathbb{N}^+$, be the arrival time of the kth claim. To derive Lundberg's generalized equation, we seek a number $z \in \mathbb{C}$ such that the process $\left\{v^{T_k} z^{-U_{T_k}},\ k \in \mathbb{N}\right\}$ forms a martingale. This is the condition that

$$\mathrm{E}\left(v^W z^{X-cW}\right) = 1. \tag{2.1}$$