Journal of Computational and Applied Mathematics 260 (2014) 494–508

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Spectral multiscale finite element for nonlinear flows in highly heterogeneous media: A reduced basis approach

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ARTICLE INFO

Article history:
Received 27 February 2013
Received in revised form 9 October 2013

Keywords:
Nonlinear permeability
Highly heterogeneous media
High contrast media
Reduced basis
Multiscale methods

ABSTRACT

In this paper, we study multiscale finite element methods for Richards' equation, a mathematical model to describe fluid flow in unsaturated and highly heterogeneous porous media. In order to compute solutions of Richard's equation, one can use numerical homogenization or multiscale methods that use two-grid procedures: a fine-grid that resolves the heterogeneities and a coarse grid where computations are done. The idea is that the coarse solution procedure captures the fine-grid variations of the solution. Since the media has complicated variations inside of coarse-grid blocks, a large error can be generated during the computation of coarse-scale solutions. In this paper, we consider the case of highly varying coefficients where variations can occur within coarse regions we develop accurate multiscale methods. In order to obtain accurate coarse-scale numerical solutions for Richards' equation, we design an effective multiscale method that is able to capture the multiscale features of the solution without discarding the small scale details. With a careful choice of the coarse basis functions for multiscale finite element methods, we can significantly reduce errors. We use coarse basis functions construction that combines local spectral problems and a Reduced Basis (RB) approach. This is an extension to the nonlinear case of the method proposed by Efendiev et al. (2012) that combines spectral constructions of coarse spaces with RB procedures to efficiently solve linear parameter dependent flow problems. The construction of coarse spaces begins with an initial choice of multiscale basis functions supported in coarse regions. These basis functions are complemented using a local, parameter dependent, weighted eigenvalue problem. The obtained basis functions can capture the small scale features of the solution within a coarse-grid block and give us an accurate coarse-scale solution. The RB procedures are used to efficiently solve for all possible flow scenarios encountered in every single iteration of a fixed point iterative method. We present representative numerical experiments.

1. Introduction

Soil water transport has considerable influence on the heat and solute transport in soils, water supply to plants, and many other important effects on the environment. Therefore, simulations of soil water flow have many applications in hydrology, agronomy, and other soil related fields. The percolation of water in the soil varies greatly according to the soil structure which has many variations depending on the location or environmental conditions. The difficulty in analyzing groundwater transport is caused mainly by the heterogeneity of subsurface formations which may span many scales. These multiple scales

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http://dx.doi.org/10.1016/j.cam.2013.10.010

dominate simulation costs and therefore we need to construct solution techniques which resolve all the small-scale effects into few larger scales.

The basic equation in the theory of groundwater flow through unsaturated porous media is Richards' equation [1],

$$D_t\theta(u) - \operatorname{div}(\kappa(x,u)\nabla(u+x_3)) = f, \quad x \in \Omega, \tag{1}$$

which describes the infiltration of water into a porous media whose pore space is filled with air and water. Here $\theta(u)$ denotes the volumetric water content, the function u represents pressure head, the nonlinear coefficient $\kappa(x,u) \geq k_0 > 0$ is the relative hydraulic conductivity and k_0 is a constant. We assume that appropriate initial and boundary data are provided. We also assume that the moisture water content $\theta(u)$ and the conductivity $\kappa(x,u)$ have functional forms depending on the pressure head u. Several functional forms which are formulated by experimental and observational data have been suggested by hydrologists and soil scientists. In this paper, we use three well-known models, namely, Haverkamp, van Genuchten, and Exponential models. See [2] and references therein.

In our research, we consider Richards' equation in the complex heterogeneous porous media which is characterized by the large variations of the conductivity, $\kappa(x,u)$. We consider the steady-state Richards' equation

$$-\operatorname{div}(\kappa(x,u)\nabla(u+x_3)) = f, \quad x \in \Omega. \tag{2}$$

The high contrast defined as the ratio between highest and lowest conductivity values brings a small scale into the problem. The scale difference makes the problem computationally expensive and often it is impossible to solve the small scale properties directly. Several approaches are proposed to solve such problems [3–6].

In this paper, we discuss multiscale finite element method (MsFEM) and its applications to the computations of Richards' equation. The main idea of MsFEM is to incorporate the small-scale information into finite element basis functions and capture their effect on the large scale via finite element computations. Originally, MsFEM is proposed for linear equations. The pre-constructed multiscale base functions interpolate a coarse-scale function which is defined at the nodal values of the coarse grid to the underlying fine grid. This idea can be generalized to nonlinear problems. In nonlinear problems, it is considered a multiscale map instead of multiscale base functions and it maps from the coarse grid space to the underlying fine grid space, see [3]. This multiscale map is constructed using the solutions of the local problems and provided us with the interpolation of the coarse-scale function, defined at the nodal values of the coarse grid to the underlying fine grid. Once the multiscale mapping is defined, we formulate the global finite element formulation of the problem.

To obtain accurate coarse-scale approximations of the solution, developing an appropriate multiscale basis functions with suitable boundary conditions is a major issue. These basis functions need to be accurate in the sense that they need to represent important features of the solution within a coarse-grid block. Several approaches to get accurate MsFEM bases were studied [7,3,8,9].

In this paper, we use local spectral basis functions which where introduced [10]. The construction of coarse spaces starts with an initial choice of multiscale basis functions that are supported in coarse regions sharing a common node. These basis functions are complemented using weighted local spectral problems that are defined in coarse blocks. The weight function in the local spectral problem is computed based on the initial choice of multiscale basis functions. In our research, we design coarse spaces based on local spectral problems using multiscale functions and we show that these coarse spaces give accurate solutions for the considered Richards' equation in heterogeneous media. By combining small-scale localizable features of the solution into initial multiscale basis functions, we show that one can achieve small dimensional coarse spaces without sacrificing the convergence properties.

Iterative procedures to solve Eq. (2) can be regarded as a parameter-dependent elliptic equation. See Section 3 for details. In this case the coefficient κ depends not only on the spatial coordinates but also on a parameter that is determined by previous approximation of the solution. Each value of the parameter gives a realization of the conductivity field. Recall that we assume that the coefficients have both small scales and high contrast. To construct local basis functions, we first find initial multiscale basis functions and construct local spectral problems for complementing the initial coarse space.

However, for nonlinear (and parameter dependent) problems, solving the local eigenvalue problem in each iteration (or for each parameter, respectively) can be expensive, especially in the cases of large size coarse blocks. Therefore we need special techniques to compute basis functions inexpensively. Here, we apply Reduced Basis (RB) approach.

The main idea of the RB method is to represent the solution of parametrized PDEs by a small set of basis functions which is referred as reduced basis. The reduced basis is constructed from a larger set of snapshots which are typically pre-computed solutions of the underlying PDE at selected parameter points. The reduced basis approach is thus adapted to the local parameter dependence of the differential operator. As a consequence, the size of the original problem can be significantly reduced since only a small number of basis functions is typically required.

The overall resulting multiscale procedure (that combines MsFEM and RB) can be framed within the Generalize Multiscale Finite Element Method (GMsFEM) recently developed in [11]. The GMsFEM method is a very general multiscale procedure that divide the computations in offline (or pre-processing) procedures and online inexpensive construction of basis functions used to approximate global solutions. In the GMsFEM offline computations some special spectral selection procedure is implemented to identify important basis. In our approach, in order to select important basis (and important information) in the off-line stage, we use RB procedures as described above. Then, in the online stage, we compute basis functions using the obtained RB-basis. We also mention that, despite of the fact that we construction basis locally, we compute coarse solutions using conforming Galerkin projections on the resulting (H^1-conforming) coarse-scale finite element spaces. A detailed