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(M, N)-coherent pairs of order (m, k) and Sobolev orthogonal polynomials

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ABSTRACT

A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-coherent pair of order (m, k) if their corresponding sequences of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy a structure relation such as

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$. In the first part of this work we focus our attention on the algebraic properties of an (M, N)-coherent pair of order (m, k). To be more precise, let us assume that $m \geq k$. If $m = k$ then $\mathcal{U}$ and $\mathcal{V}$ are related by a rational factor (in the distributional sense); if $m > k$ then $\mathcal{U}$ and $\mathcal{V}$ are semiclassical and they are again related by a rational factor. In the second part of this work we deal with a Sobolev type inner product defined in the linear space of polynomials with real coefficients, $\mathbb{P}$, as

$$\langle p(x), q(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)d\mu_1(x), \quad p, q \in \mathbb{P},$$

where λ is a positive real number, m is a positive integer number and (μ_0, μ_1) is a (M, N)-coherent pair of order m of positive Borel measures supported on an infinite subset of the real line, meaning that the sequences of monic orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ with respect to μ_0 and μ_1, respectively, satisfy a structure relation as above with $k = 0$, $a_{i,n}$ and $b_{i,n}$ being real numbers fulfilling the above mentioned conditions. We generalize several recent results known in the literature in the framework of Sobolev orthogonal polynomials and their connections with coherent pairs (introduced in [A. Iserles, P.E. Koch, S.P. Nørsett, J.M. Sanz-Serna, On polynomials orthogonal with respect to certain Sobolev inner products J. Approx. Theory 65 (2) (1991) 151–175]) and their extensions. In particular, we show how to compute the coefficients of the Fourier expansion of functions on an appropriate Sobolev space (defined by the above inner product) in terms of the sequence of Sobolev orthogonal polynomials $\{S_n(x; \lambda)\}_{n\geq 0}$.

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1. Introduction

In this work we deal with sequences of monic polynomials, $\{S_n(x;\lambda)\}_{n\geq 0}$, orthogonal with respect to the Sobolev inner product

$$\langle p(x), q(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)d\mu_1(x), \quad p, q \in \mathbb{P}, \tag{1.1}$$

where λ is a positive real number, m is a positive integer number (it indicates a derivative) and (μ_0, μ_1) is a (M, N)-coherent pair of order m of positive Borel measures supported on an infinite subset of the real line, i.e., if $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are the sequences of monic orthogonal polynomials (SMOPs) with respect to μ_0 and μ_1, respectively, then

$$\sum_{i=0}^{M} a_{i,n}P^{(m)}_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n}Q_{n-i}(x), \quad n \geq 0, \tag{1.2}$$

where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$.

The case $(M, N) = (1, 0)$ and $m = 1$ has a special historical importance. Such a pair of measures is said to be a coherent pair and it has been introduced and analyzed by A. Iserles, P.E. Koch, S.P. Nørsett, and J.M. Sanz-Serna [1]. Later on, F. Marcellán and J. Petronilho [2] described all the coherent pairs of measures when one of the measures is a classical one. Finally, H.G. Meijer [3] proved that there are no other coherent pairs, showing that, indeed, in a coherent pair one of the measures must be a classical one (Jacobi or Laguerre) and the other one is a rational perturbation of it. Meijer's paper had a great influence in the subsequent developments of the theory of coherent pairs of orthogonal polynomials. Indeed, after these works, several other ones appeared in the literature, dealing with generalizations of the notion of coherence, including in a more general framework of quasi-definite linear functionals. For instance, among others, we mention here the works by K.H. Kwon, J.H. Lee, and F. Marcellán [4], F. Marcellán, A. Martínez-Finkelshtein and J.J. Moreno-Balcázar [5], M. de Bruin and H.G. Meijer [6], M. Alfaro, F. Marcellán, A. Peña, and M.L. Rezola [7–10], A.M. Delgado and F. Marcellán [11,12], J. Petronilho [13], M.N. de Jesus and J. Petronilho [14,15], A. Branquinho and M.N. Rebocho [16], F. Marcellán and N.C. Pinzón-Cortés [17], and M. Alfaro, A. Peña, J. Petronilho, and M.L. Rezola [18]. For a review about these and other contributions, see e.g. the introductory sections in the recent papers [15,17].

All these generalizations of the notion of coherence may be regarded as special cases of the notion of (M, N)-coherence of order (m, k) to be considered in the present paper. Indeed, given two SMOPs $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$, and four nonnegative integer numbers M, N, m, k, we say these two SMOPs form a (M, N)-coherent pair of order (m, k) if a relation such as

$$\sum_{i=0}^{M} a_{i,n}P^{(m)}_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n}Q^{(k)}_{n+k-i}(x), \quad n \geq 0,$$

holds, where $a_{i,n}$ and $b_{i,n}$ are complex numbers such that $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$. The above structure relation has been already considered in [14], where it has been proved that, under some natural conditions, and assuming, without lost of generality, that $0 \leq k \leq m$, the regular moment linear functionals $\mathcal{U}$ and $\mathcal{V}$ associated with the SMOPs $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ (respectively) fulfill a distributional differential equation

$$D^{m-k}(\phi(x)\mathcal{V}) = \psi(x)\mathcal{U},$$

where $\phi(x)$ and $\psi(x)$ are some polynomials. Furthermore in [14] the authors also proved that if $m = k$ then $\mathcal{U}$ and $\mathcal{V}$ are related by a rational factor and, if $m = k + 1$, then both $\mathcal{U}$ and $\mathcal{V}$ must be semiclassical, being also related by a rational factor. For a survey about the theory of semi-classical linear functionals, the basic reference is P. Maroni [19].

When $m > k + 1$ the problem of determining whether $\mathcal{U}$ and $\mathcal{V}$ are semiclassical (for arbitrary M and N) remained open. In the present work we fill this gap by proving that even when $m > k + 1$ both $\mathcal{U}$ and $\mathcal{V}$ are semiclassical and they are related by a rational factor. This will be stated in Section 3. On the other hand, when the above linear functionals are associated with positive Borel measures, then a useful algebraic relation between the sequences $\{S_n(x;\lambda)\}_{n\geq 0}$ and $\{P_n(x)\}_{n\geq 0}$ will be deduced, provided that the measures form an (M, N)-coherent pair of order m in the sense of (1.2) and $\{S_n(x;\lambda)\}_{n\geq 0}$ is an SMOP with respect to the inner product (1.1). This will be the topic to be analyzed in Section 4. Notice that an inner product of this type, involving higher order derivatives, was already considered in [17] in a situation corresponding to $(1, 1)$-coherence of order m. In Section 5 we built and implement an efficient algorithm for the computation of the Fourier–Sobolev coefficients, i.e., the coefficients of the Fourier expansion of functions of the Sobolev space $W^{m,2}(I, \mu_0, \mu_1)$ in terms of the SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$, thus extending to the more general framework of (M, N)-coherence of order m the previous algorithms known in the literature for coherence [1], generalized coherence [4], and (M, N)-coherence (of order 1) [15]. Notice that from such an algorithm the evaluation of the Fourier–Sobolev coefficients does not need the explicit expressions of the Sobolev orthogonal polynomials. This is an extension of a remarkable fact pointed out by Iserles et al. in [20] for coherent pairs. In such a paper the authors point out that when we wish to approximate a function by its projection into the linear space of polynomials and, simultaneously, to approximate its derivative by the derivative of the polynomial approximant in the linear space $L^2([-1, 1]; dx)$ the Fourier–Sobolev projector in the Sobolev space $W^{1,2}([-1, 1]; dx, dx)$is more valuable than the standard Fourier projector in such a space. Given that the derivative of the function is steep, it is only expected that the