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Some results on a starlike log-harmonic mapping of order alpha

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ABSTRACT

Let $H(D)$ be the linear space of all analytic functions defined on the open unit disc $D = z \in C : |z| < 1$. A sense preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation

$$f_z = w(z) f_z (f_z / f)$$

where $w(z) \in H(D)$ is the second dilatation of f such that $|w(z)| < 1$ for all $z \in D$.

A sense preserving log-harmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\frac{f_{\bar{z}}}{\bar{f}} = w(z). \frac{f_z}{f} \tag{0.1}$$

where $w(z)$ the second dilatation of f and $w(z) \in H(D)$, $|w(z)| < 1$ for every $z \in D$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)} \tag{0.2}$$

where $h(z)$ and $g(z)$ are analytic in D with the normalization $h(0) \neq 0, g(0) = 1$. On the other hand if f vanishes at $z = 0$, but it is not identically zero, then f admits the following representation

$$f(z) = z.z^{2\beta} h(z)\overline{g(z)} \tag{0.3}$$

where Re $\beta > -\frac{1}{2}$, $h(z)$ and $g(z)$ are analytic in the open disc D with the normalization $h(0) \neq 0, g(0) = 1$ (Abdulhadi and Bshouty, 1988) [2], (Abdulhadi and Hengartner, 1996) [3].

In the present paper, we will give the extent of the idea, which was introduced by Abdulhadi and Bshouty (1988) [2]. One of the interesting applications of this extent idea is an investigation of the subclass of log-harmonic mappings for starlike log-harmonic mappings of order alpha.

1. Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Next, denote by $\mathcal{P}(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

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regular in $\mathbb{D}$, such that $p(z)$ is in $\mathcal{P}(A,B)$ if and only if

$$p(z)=\frac{1+A\phi(z)}{1+B\phi(z)},\quad -1\leq B<A\leq 1 \tag{1.1}$$

for some function $\phi(z)\in\Omega$ and for every $z\in\mathbb{D}$. Therefore we have $p(0)=1$, $\operatorname{Re}p(z)>\frac{1-A}{1-B}>0$ whenever $p(z)\in\mathcal{P}(A,B)$. Moreover, let $\mathcal{S}^*(A,B)$ denote the family of functions

$$s(z)=z+a_2z^2+\cdots$$

regular in $\mathbb{D}$, and such that $s(z)$ is in $\mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(z\frac{s'(z)}{s(z)}\right)=p(z)=\frac{1+\phi(z)}{1-\phi(z)},\quad p(z)\in\mathcal{P}(1,-1). \tag{1.2}$$

Let $S_1(z)$ and $S_2(z)$ be analytic functions in $\mathbb{D}$ with $S_1(0)=S_2(0)$. We say that $S_1(z)$ is subordinate to $S_2(z)$ and denoted by $S_1(z)\prec S_2(z)$, if $S_1(z)=S_2(\phi(z))$ for some function $\phi(z)\in\Omega$ and every $z\in\mathbb{D}$. If $S_1(z)\prec S_2(z)$, then $S_1(\mathbb{D})\subset S_2(\mathbb{D})$ [1].

The radius of starlikeness of the class of sense-preserving log-harmonic mapping is

$$r_s=\sup\left\{r|\operatorname{Re}\left(\frac{zf_z-\bar{z}f_{\bar{z}}}{f}\right)>0,0<r<1\right\}.$$

Finally, let $H(D)$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D}$. A sense-preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f_{\bar{z}}}}{\overline{f}}=w(z)\frac{f_z}{f}, \tag{1.3}$$

where $w(z)\in H(\mathbb{D})$ is the second dilatation of f such that $|w(z)|<1$ for every $z\in\mathbb{D}$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f=h(z)\overline{g(z)} \tag{1.4}$$

where $h(z)$ and $g(z)$ are analytic functions in $\mathbb{D}$.

On the other hand, if f vanishes at $z=0$ and at no other point, then f admits the representation,

$$f=z\,|z|^{2\beta}\,h(z)\overline{g(z)}, \tag{1.5}$$

where $\operatorname{Re}\beta>-1/2$, $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with $g(0)=1$ and $h(0)\neq 0$. We note that the class of log-harmonic mappings is denoted by $\mathcal{S}_{LH}$.

Let $f=zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{LH}$. We say that f is a Janowski starlike log-harmonic mapping if

$$1+\frac{1}{b}\left(\frac{zf_z-\bar{z}f_{\bar{z}}}{f}-1\right)=p(z)=\frac{1+A\phi(z)}{1+B\phi(z)},\quad p(z)\in\mathcal{P}(A,B) \tag{1.6}$$

where $-1\leq B<A\leq 1$, $b\neq 0$ and complex and denote by $\mathcal{S}^*_{LH}(A,B,b)$ the set of all starlike log-harmonic mappings. Also we denote by $\mathcal{S}^*_{PLH}(A,B,b)$ the class of all functions in $\mathcal{S}^*_{LH}(A,B,b)$ for which $(zh(z))\in\mathcal{S}^*(A,B)$ for all $z\in\mathbb{D}$.

We note that if we give special values to b, then we obtain important subclasses of Janowski starlike log-harmonic mappings

i. For $b=0$, we obtain the class of starlike log-harmonic mappings.
ii. For $b=1-\alpha$, $0\leq\alpha<1$, we obtain the class of starlike log-harmonic mappings of order α.
iii. For $b=e^{-i\lambda}\cos\lambda$, $|\lambda|<\frac{\pi}{2}$, we obtain the class of λ-spirallike log-harmonic mappings.
iv. For $b=(1-\alpha)e^{-i\lambda}\cos\lambda$, $0\leq\alpha<1$, $|\lambda|<\frac{\pi}{2}$, we obtain the class of λ-spirallike log-harmonic mappings of order α.

2. Main results

Theorem 2.1. *Let $f=zh(z)\overline{g(z)}$ be an element of $\mathcal{S}^*_{LH}(A,B,b)$. Then*

$$f=zh(z)\overline{g(z)}\in\mathcal{S}^*_{LH}(A,B,b)\Leftrightarrow\begin{cases} z\dfrac{h'(z)}{h(z)}-\overline{z}\dfrac{\overline{g'(z)}}{\overline{g(z)}}\prec\dfrac{b(A-B)z}{1+Bz}; & B\neq 0,\\[2ex] z\dfrac{h'(z)}{h(z)}-\overline{z}\dfrac{\overline{g'(z)}}{\overline{g(z)}}\prec bAz; & B=0.\end{cases} \tag{2.1}$$